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GENERALIZATION OF CERTAIN CONTINUED FRACTIONS OF SRINIVASA RAMANUJAN

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Abstract: In this paper we have established certain continued fraction representation for the ratio of abnormal basic hypergeometric series with some of its contiguous functions. We have also obtained some interesting special cases of our continued fraction expansions which generalize some famous q-continued fraction identities of Srinivasa Ramanujan.

Keywords and Phrases: Abnormal basic hypergeometric series, basic hypergeometric series, q-series and continued fractions.

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1. Introduction, Notations and Definitions

Continued fraction has been centre of attraction for applied mathematicians as well as pure mathematicians of previous centuries. In previous centuries there are so many results, which are established in terms of continued fraction. It is also a tool which acts as a bridge between pure and applied mathematicians. So, the attraction of continued fraction for today's mathematicians has also amplified. Ramanujan, a pioneer in the theory of continued fractions has recorded scores of continued fraction identities in chapter 12 of his second notebook [11] and in his lost notebook [3]. Methods used by the great Indian mathematician, Srinivasa Ramanujan, to obtain many of his fascinating results remain a mystery. This part of Ramanujan's work has been treated and developed by several authors including Andrews and Bowman [4], Hirschhorn [9], Adiga [1,2], Denis and Singh [5,6,7,8] and Somashekhra et al. [12] etc. In a recent communication Mishra et al. [10] and Srivastava, Singh and Singh [13] have given some interesting continued fraction expansions of q-series, which also generalize some of the known results of Andrews and Berndt [4].

The purpose of this paper is to establish certain continued fraction representation for the ratio of abnormal basic hypergeometric series which also establish the generalization of certain continued fraction identities due to Srinivasa Ramanujan [11].

We shall use the following definitions and notations throughout the paper.
An abnormal basic hypergeometric series is defined as

$$ {}_A\phi_B\left[\begin{array}{l} a_1, a_2, ..., a_A; x \\ b_1, b_2, ..., b_B; q^\lambda \end{array}\right] = \sum_{n\geq 0} \frac{[a_1]_n[a_2]_n...[a_A]_n x^n q^{\lambda n(n+1)/2}}{[b_1]_n[b_2]_n...[b_B]_n[q]_n} \tag{1.1}$$

for $|x| < \infty$ if $\lambda > 0$ and for $|q|, |x| < 1$ if $\lambda = 0$.
Also, for $|q| < 1$ and arbitrary a
$[a]_n \equiv (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), \quad n > 0$
and
$[a]_0 = 1$.

2. Main Results

In this paper we shall establish the following set of results,

$$ \frac{{}_1\phi_1\left[\begin{array}{l} a; xq \\ b; q \end{array}\right]}{{}_1\phi_1\left[\begin{array}{l} a; x \\ b; q \end{array}\right]} = \frac{1}{1+}\frac{(1-a)xq}{1+}\frac{-(axq^2+b)}{1+}\frac{(1-aq)xq^2}{1+...} \tag{2.1}$$

$$ = \frac{1}{1+}\frac{(1-a)xq}{1-b-xq+}\frac{(1-aq)xq}{1-bq-xq+}\frac{(1-aq^2)xq}{1-bq^2-xq+...} \tag{2.2}$$

$$ = \frac{1}{(1+b/q+xq)-}\frac{axq+b/q}{(1+b/q+xq^2)-}\frac{axq^2+b/q}{(1+b/q+xq^3)-...} \tag{2.3}$$

$$ = \frac{1}{1+xq+}\frac{(b-a)xq}{1-b+xq^2+}\frac{(bq-a)xq^2}{1-bq+xq^3+...} \tag{2.4}$$

3. Proof of (2.1) to (2.4)

It is easily verified that

$$ {}_1\phi_1\left[\begin{array}{l} a; x \\ b; q \end{array}\right] - {}_1\phi_1\left[\begin{array}{l} a/q; xq \\ b; q \end{array}\right] = \frac{xq}{1-b}\,{}_1\phi_1\left[\begin{array}{l} a; xq \\ bq; q \end{array}\right] \tag{3.1}$$

$$ {}_1\phi_1\left[\begin{array}{l} a; x \\ b; q \end{array}\right] - {}_1\phi_1\left[\begin{array}{l} aq; x \\ b; q \end{array}\right] = \frac{-axq}{1-b}\,{}_1\phi_1\left[\begin{array}{l} aq; xq \\ bq; q \end{array}\right] \tag{3.2}$$

$$ {}_1\phi_1\left[\begin{matrix} a;x \\ b;q \end{matrix}\right] - {}_1\phi_1\left[\begin{matrix} a;x \\ bq;q \end{matrix}\right] = -b\ {}_1\phi_1\left[\begin{matrix} a;xq \\ bq;q \end{matrix}\right] \tag{3.3} $$

Replacing a by aq in (3.1) and adding to (3.2) we get

$$ {}_1\phi_1\left[\begin{matrix} a;x \\ b;q \end{matrix}\right] - {}_1\phi_1\left[\begin{matrix} a;xq \\ b;q \end{matrix}\right] = \frac{(1-a)xq}{1-b}\ {}_1\phi_1\left[\begin{matrix} aq;xq \\ bq;q \end{matrix}\right] \tag{3.4} $$

Replacing a by aq in (3.3), multiplying (3.2) by $(1-b)$ and adding these two equations we get

$$ (1-b){}_1\phi_1\left[\begin{matrix} a;x \\ b;q \end{matrix}\right] - {}_1\phi_1\left[\begin{matrix} aq;x \\ bq;q \end{matrix}\right] = -(axq+b)\ {}_1\phi_1\left[\begin{matrix} aq;xq \\ bq;q \end{matrix}\right]. \tag{3.5} $$

Changing a by aq^i, b by bq^i and x by xq^i in (3.4), a by aq^i, b by bq^i and x by xq^{i+1} in (3.5) and after some simplification we get

$$ \frac{{}_1\phi_1\left[\begin{matrix} aq^i;xq^i \\ bq^i;q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq^i;xq^{i+1} \\ bq^i;q \end{matrix}\right]} = 1 + \cfrac{(1-aq^i)xq^{i+1}}{(1-bq^i)\times\cfrac{{}_1\phi_1\left[\begin{matrix} aq^i;xq^{i+1} \\ bq^i;q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq^{i+1};xq^{i+1} \\ bq^{i+1};q \end{matrix}\right]}} \tag{3.6} $$

$$ (1-bq^i)\frac{{}_1\phi_1\left[\begin{matrix} aq^i;xq^{i+1} \\ bq^i;q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq^{i+1};xq^{i+1} \\ bq^{i+1};q \end{matrix}\right]} = 1 + \cfrac{-(axq^{2i+2}+bq^i)}{\cfrac{{}_1\phi_1\left[\begin{matrix} aq^{i+1};xq^{i+1} \\ bq^{i+1};q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq^{i+1};xq^{i+2} \\ bq^{i+1};q \end{matrix}\right]}} \tag{3.7} $$

Taking $i=0$ in (3.6) we have

$$ \frac{{}_1\phi_1\left[\begin{matrix} a;x \\ b;q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} a;xq \\ b;q \end{matrix}\right]} = 1 + \cfrac{(1-a)xq}{(1-b)\cfrac{{}_1\phi_1\left[\begin{matrix} a;xq \\ b;q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq;xq \\ bq;q \end{matrix}\right]}} $$

(Using (3.7) with $i = 0$)

$$= 1 + \cfrac{(1-a)xq}{1 - \cfrac{axq^2 + b}{\cfrac{{}_1\phi_1\left[\begin{matrix} aq; xq \\ bq; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq; xq^2 \\ bq; q \end{matrix}\right]}}}$$

$$= \frac{1}{1+}\,\frac{(1-a)xq}{1+}\,\frac{-(axq^2+b)}{1+}\,\frac{(1-aq)xq^2}{1+\ldots}$$

(Using alternately (3.6) and (3.7) with $i = 1, 2, \ldots$). This proves (2.1).
Again replacing a by aq and b by bq in (3.1), b by bq in (3.2), taking the negative of (3.1) and adding these three equations in (3.3) we have

$$(1-b)\,{}_1\phi_1\left[\begin{matrix} a/q; xq \\ b; q \end{matrix}\right] = (1-b-xq)\,{}_1\phi_1\left[\begin{matrix} a; xq \\ bq; q \end{matrix}\right] + \frac{(1-a)xq}{1-bq}\,{}_1\phi_1\left[\begin{matrix} aq; xq \\ bq^2; q \end{matrix}\right] \tag{3.8}$$

Next, replacing a by aq^{i+1}, b by bq^i in (3.8) and after some simplification we have

$$(1-bq^i)\frac{{}_1\phi_1\left[\begin{matrix} aq^i; xq \\ bq^i; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq^{i+1}; xq \\ bq^{i+1}; q \end{matrix}\right]} = (1-bq^i-xq) + \cfrac{(1-aq^{i+1})xq}{(1-bq^{i+1}) \times \cfrac{{}_1\phi_1\left[\begin{matrix} aq^{i+1}; xq \\ bq^{i+1}; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq^{i+2}; xq \\ bq^{i+2}; q \end{matrix}\right]}} \tag{3.9}$$

Further, taking $i = 0$ in (3.9) and using in (3.4) we have

$$\frac{{}_1\phi_1\left[\begin{matrix} a; x \\ b; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} a; xq \\ b; q \end{matrix}\right]} = 1 + \cfrac{(1-a)xq}{(1-b-xq) + \cfrac{(1-aq)xq}{(1-bq) \times \cfrac{{}_1\phi_1\left[\begin{matrix} aq; xq \\ bq; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} aq^2; xq \\ bq^2; q \end{matrix}\right]}}} \tag{3.10}$$

Repeated application of (3.9) with $i = 1, 2, ...$ in (3.10) yields,

$$\frac{{}_1\phi_1\left[\begin{matrix} a; xq \\ b; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} a; x \\ b; q \end{matrix}\right]} = \frac{1}{1+}\,\frac{(1-a)xq}{1-b-xq+}\,\frac{(1-aq)xq}{1-bq-xq+}\,\frac{(1-aq^2)xq}{1-bq^2-xq+...}$$

This proves (2.2).

Changing a to a/q and x by xq in (3.2), x to xq in (3.3), taking the negative of (3.3) and adding these three equations in (3.1) we get

$$ {}_1\phi_1\left[\begin{matrix} a; x \\ bq; q \end{matrix}\right] = (1+b+xq)\ {}_1\phi_1\left[\begin{matrix} a; xq \\ bq; q \end{matrix}\right] - (axq+b)\ {}_1\phi_1\left[\begin{matrix} a; xq^2 \\ bq; q \end{matrix}\right] \tag{3.11}$$

Next, replacing x by xq^i and b by b/q in (3.11) we have

$$\frac{{}_1\phi_1\left[\begin{matrix} a; xq^i \\ b; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} a; xq^{i+1} \\ b; q \end{matrix}\right]} = \left(1+\frac{b}{q}+xq^{i+1}\right) - \frac{axq^{i+1}+b/q}{\dfrac{{}_1\phi_1\left[\begin{matrix} a; xq^{i+1} \\ b; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} a; xq^{i+2} \\ b; q \end{matrix}\right]}} \tag{3.12}$$

Repeated application of (3.12) with $i = 0, 1, 2, ...$ yields, after simplification

$$\frac{{}_1\phi_1\left[\begin{matrix} a; xq \\ b; q \end{matrix}\right]}{{}_1\phi_1\left[\begin{matrix} a; x \\ b; q \end{matrix}\right]} = \frac{1}{(1+b/q+xq)-}\,\frac{axq+b/q}{(1+b/q+xq^2)-}\,\frac{axq^2+b/q}{(1+b/q+xq^3)-...}$$

This proves (2.3).

Changing x to xq, b to bq in (3.1) and multiplying the result by b; a to a/q, x to xq and b to bq in (3.2); a to a/q and x to xq in (3.3); taking the negative of (3.3) and adding all these four equations to (3.2) we have

$$ {}_1\phi_1\left[\begin{matrix} a; x \\ bq; q \end{matrix}\right] = (1+xq)\ {}_1\phi_1\left[\begin{matrix} a; xq \\ bq; q \end{matrix}\right] + \frac{(bq-a)xq}{1-bq}\ {}_1\phi_1\left[\begin{matrix} a; xq^2 \\ bq^2; q \end{matrix}\right] \tag{3.13}$$

Replacing x by xq in (3.13); x by xq in (3.3) and adding these two equations we have

$$(1-b)\ {}_1\phi_1\left[\begin{matrix} a; xq \\ b; q \end{matrix}\right] = (1-b+xq^2)\ {}_1\phi_1\left[\begin{matrix} a; xq^2 \\ bq; q \end{matrix}\right] + \frac{(bq-a)xq^2}{1-bq}\ {}_1\phi_1\left[\begin{matrix} a; xq^3 \\ bq^2; q \end{matrix}\right] \tag{3.14}$$

Again, replacing b by bq^i and x by xq^i in (3.14) we have

$$(1-bq^i)\frac{{}_1\phi_1\left[\begin{array}{c} a; xq^{i+1} \\ bq^i; q \end{array}\right]}{{}_1\phi_1\left[\begin{array}{c} a; xq^{i+2} \\ bq^{i+1}; q \end{array}\right]} = (1-bq^i+xq^{i+2}) + \frac{(bq^{i+1}-a)xq^{i+2}}{(1-bq^{i+1}) \times \dfrac{{}_1\phi_1\left[\begin{array}{c} a; xq^{i+2} \\ bq^{i+1}; q \end{array}\right]}{{}_1\phi_1\left[\begin{array}{c} a; xq^{i+3} \\ bq^{i+2}; q \end{array}\right]}} \tag{3.15}$$

Changing b to b/q in (3.13) we have

$$\frac{{}_1\phi_1\left[\begin{array}{c} a; x \\ b; q \end{array}\right]}{{}_1\phi_1\left[\begin{array}{c} a; xq \\ b; q \end{array}\right]} = (1+xq) + \frac{(b-a)xq}{(1-b) \times \dfrac{{}_1\phi_1\left[\begin{array}{c} a; xq \\ b; q \end{array}\right]}{{}_1\phi_1\left[\begin{array}{c} a; xq^2 \\ bq; q \end{array}\right]}} \tag{3.16}$$

Repeated application of (3.15) with $i = 0, 1, 2, ...$ in (3.16) yields

$$\frac{{}_1\phi_1\left[\begin{array}{c} a; x \\ b; q \end{array}\right]}{{}_1\phi_1\left[\begin{array}{c} a; xq \\ b; q \end{array}\right]} = (1+xq) + \frac{(b-a)xq}{1-b+xq^2+}\,\frac{(bq-a)xq^2}{1-bq+xq^3+...},$$

or

$$\frac{{}_1\phi_1\left[\begin{array}{c} a; xq \\ b; q \end{array}\right]}{{}_1\phi_1\left[\begin{array}{c} a; x \\ b; q \end{array}\right]} = \frac{1}{1+xq+}\,\frac{(b-a)xq}{1-b+xq^2+}\,\frac{(bq-a)xq^2}{1-bq+xq^3+...}$$

This proves (2.4).
This completes the proof of (2.1)-(2.4).

4. Special Cases

In this section we shall discuss some of very interesting special cases of our result.

If we replace a by $-\lambda/a$, b by $-bq$ and x by a in (2.1)-(2.4) we obtain the following results due to Adiga [1, (Theo. 1-4), p. 31-34]

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{[-\lambda/a]_n q^{n(n+1)/2}(aq)^n}{[-bq]_n[q]_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{[-\lambda/a]_n q^{n(n+1)/2}(a)^n}{[-bq]_n[q]_n}}$$

$$= \frac{1}{1+} \frac{(aq+\lambda q)}{1+} \frac{(bq+\lambda q^2)}{1+\ldots} \frac{(aq^{n+1}+\lambda q^{2n+1})}{1+} \frac{(bq^{n+1}+\lambda q^{2n+2})}{1+\ldots}, \tag{4.1}$$

$$= \frac{1}{1+} \frac{aq+\lambda q^2}{1-aq+bq+} \frac{aq+\lambda q^2}{1-aq+bq^2+\ldots} \frac{aq+\lambda q^n}{1-aq+bq^n+\ldots}, \tag{4.2}$$

$$= \frac{1}{1-b+aq+} \frac{b+\lambda q}{1-b+aq^2+\ldots} \frac{b+\lambda q^n}{1-b+aq^{n+1}+\ldots}, \tag{4.3}$$

$$= \frac{1}{1+aq+} \frac{\lambda q - abq^2}{1+q(aq+b)+\ldots} \frac{\lambda q^n - abq^{2n}}{1+q^n(aq+b)+\ldots}. \tag{4.4}$$

Taking $a \to 0$ in (4.1)-(4.3) we obtain the following results of Ramanujan [11, (Entry 15, 16), Ch. 16, p. 196]

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}\lambda^n}{[-bq]_n[q]_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}\lambda^n}{[-bq]_n[q]_n}} = \frac{1}{1+} \frac{\lambda q}{1+} \frac{\lambda q^2}{1+\ldots} \frac{\lambda q^{n+1}}{1+\ldots}, \tag{4.5}$$

$$= \frac{1}{1+} \frac{\lambda q}{1+bq+} \frac{\lambda q^2}{1+bq^2+\ldots} \frac{\lambda q^n}{1+bq^n+\ldots}, \tag{4.6}$$

$$= \frac{1}{1-b+} \frac{b-\lambda q}{1-b+\ldots} \frac{b+\lambda q^n}{1-b+\ldots} \tag{4.7}$$

A number of other special cases could also be deduced.

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