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CERTAIN DOUBLE SERIES ROGERS - RAMANUJAN TYPE IDENTITIES

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Abstract: This paper contains certain double series Rogers- Ramanujan type identities which are derived as special cases of an application of Bailey's transform.

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1. Introduction

Throughout this paper we shall adopt certain notation and definitions which are stated below. Let α, β and q be complex numbers and $|q| < 1$, then

$$(\alpha; q)_n = (1-\alpha)(1-\alpha q)...(1-\alpha q^{n-1}), \quad n = 1, 2, 3, ... \tag{1.1}$$

$$(\beta; q)_\infty = \prod_{k=1}^{\infty}(1-\beta q^k), \tag{1.2}$$

and

$$(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m; q)_n = (\alpha_1; q)_n(\alpha_2; q)_n(\alpha_3; q)_n...(\alpha_m; q)_n \tag{1.3}$$

With the above notations we define basic hypergeometric series as:

$${}_r\Phi_s\left[\begin{array}{c}\alpha_1, \alpha_2, ..., \alpha_r\\ \beta_1, \beta_2, ..., \beta_s\end{array}; q; z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1, \alpha_2, ..., \alpha_r; q)_n z^n}{(\beta_1, \beta_2, ..., \beta_s; q)_n}\left\{(-1)^n q^{n(n-1)/2}\right\}^{1+n-r} \tag{1.4}$$

Bailey [2,3] stated a lemma as follows:
If
$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.5}$$
and
$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \tag{1.6}$$
then under suitable conditions of convergence
$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.7}$$
where $\alpha_r, \delta_r, u_r, v_r$ are arbitrary sequences of r.
If we substitute the value of β_n from (1.5) in the above equation and then using the following identity [10; Lemma 1(2), p. 100]
$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} A(n,r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A(n+r,r) \tag{1.8}$$
then (1.7) takes the form
$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n,r=0}^{\infty} \alpha_r u_n \delta_{n+r} v_{n+2r} \tag{1.9}$$

2. Main Results

Theorem 2.1. *Let us take* $u_r = \frac{1}{(q^2;q^2)_r}, v_r = \frac{1}{(aq;q^2)_r}$ *then*
$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q^2;q^2)_{n-r}(aq;q^2)_{n+r}} \tag{2.1}$$
and
$$\gamma_n = \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{(q^2;q^2)_r(aq;q^2)_{2n+r}}. \tag{2.2}$$
Here, we shall establish following result
$$\frac{1}{(aq;q^2)_\infty}\sum_{n=0}^{\infty} a^n q^{2n^2-n}\alpha_n = \sum_{n,r=0}^{\infty} \frac{a^{n+r}q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_n(aq;q^2)_{n+2r}}\alpha_r \tag{2.3}$$

Proof. On substituting the above values of β_n, γ_n in (1.9) and applying the identity (1.8) we get

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{\alpha_r \delta_{n+r}}{(q^2;q^2)_n (aq;q^2)_{n+2r}} \tag{2.4}$$

Assuming $\delta_r = (\rho_1, \rho_2; q^2)_r \left(\frac{aq}{\rho_1\rho_2}\right)^r$ such that $\rho_1, \rho_2 \neq 0$ and substituting it in (2.2), we get,

$$\gamma_n = \sum_{r=0}^{\infty} \frac{(\rho_1, \rho_2; q^2)_{r+n} \left(\frac{aq}{\rho_1\rho_2}\right)^{n+r}}{(q^2;q^2)_r (aq;q^2)_{2n+r}} \tag{2.5}$$

and further simplifying the equation (2.5) reduces as follows,

$$\gamma_n = \frac{(\rho_1, \rho_2; q^2)_n}{(aq;q^2)_{2n}} \left(\frac{aq}{\rho_1\rho_2}\right) {}_2\Phi_1 \left[\begin{matrix} \rho_1 q^{2n}, \rho_2 q^{2n} \\ aq^{1+4n} \end{matrix} ; q^2, \frac{aq}{\rho_1\rho_2} \right] \tag{2.6}$$

Using the identity ${}_2\Phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q, \frac{c}{ab} \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}$ we get

$$\gamma_n = \frac{(\rho_1, \rho_2; q^2)_n}{(aq;q^2)_{2n}} \left(\frac{aq}{\rho_1\rho_2}\right)^n \frac{\left(\frac{aq^{1+2n}}{\rho_1}, \frac{aq^{1+2n}}{\rho_2}; q^2\right)_\infty}{\left(aq^{1+4n}, \frac{aq}{\rho_1\rho_2}; q^2\right)_\infty} \tag{2.7}$$

Substituting the values of γ_n and δ_n in (2.4) we finally get,

$$\sum_{n=0}^{\infty} \frac{\alpha_n (\rho_1, \rho_2; q^2)_n}{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q^2\right)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \frac{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q^2\right)_\infty}{\left(aq, \frac{aq}{\rho_1\rho_2}; q^2\right)_\infty}$$
$$= \sum_{n,r=0}^{\infty} \frac{\alpha_r (\rho_1, \rho_2; q^2)_{n+r}}{(q^2;q^2)_n (aq;q^2)_{n+2r}} \left(\frac{aq}{\rho_1\rho_2}\right)^{n+r} \tag{2.8}$$

Assuming that $\rho_1, \rho_2 \to \infty$ then (2.8) finally yields,

$$\frac{1}{(aq;q^2)_\infty} \sum_{n=0}^{\infty} a^n q^{2n^2-n} \alpha_n = \sum_{n,r=0}^{\infty} \frac{a^{n+r} q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_n (aq;q^2)_{n+2r}} \alpha_r,$$

which is precisely (2.3).

3. Special Cases of Theorem 2.1.

In this section we shall formulate certain identities as special cases of (2.3)

(i) Substituting $\alpha_n = \frac{q^n}{(q^2;q^2)_n(-q;q)_{2n}}$ in (2.3),

$$\frac{1}{(aq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{a^n q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \sum_{n,r=0}^{\infty}\frac{a^{n+r}q^{2(n+r)^2-n}}{(q^2;q^2)_r(-q;q)_{2r}(q^2;q^2)_n(aq;q^2)_{n+2r}} \tag{3.1}$$

(ii) Let $a = 1$ in (3.1) we get,

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2-n}}{(q^2;q^2)_r(-q;q)_{2r}(q^2;q^2)_n(q;q^2)_{n+2r}}$$

Using [8, (33)] then above equation yields,

$$(q;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2-n}}{(q^2;q^2)_n(q;q^2)_{n+2r}(-q;q)_{2r}(q^2;q^2)_r} = \frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{3.2}$$

(iii) Let $a = q^2$ in (3.1) we get

$$\frac{1}{(q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n}q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \sum_{n,r=0}^{\infty}\frac{q^{2(n+r)}q^{2(n+r)^2-n}}{(q^2;q^2)_r(-q;q)_{2r}(q^2;q^2)_n(q^3;q^2)_{n+2r}}$$

Using [8, (32)] we get,

$$(q^3;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2+(n+r)+r}}{(q^2;q^2)_r(-q;q)_{2r}(q^2;q^2)_n(q^3;q^2)_{n+2r}} = \frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{3.3}$$

(iv) Substituting $\alpha_n = \frac{q^n}{(q^2;q^2)_n}$ in (2.3),

$$\frac{1}{(aq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{a^n q^{2n^2}}{(q^2;q^2)_n} = \sum_{n,r=0}^{\infty}\frac{a^{n+r}q^{2(n+r)^2-n}}{(q^2;q^2)_r(q^2;q^2)_n(aq;q^2)_{n+2r}}$$

Let $a = 1$ and using [1, (10.1.1) p. 241] in the above equation we get,

$$(q;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2-n}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r} = \frac{1}{(q^2,q^4;q^5)_\infty} \tag{3.4}$$

(v) Let $a = q^3, \alpha_n = 1$ in (2.3) we get,

$$\frac{1}{(q^4;q^2)_\infty}\sum_{n=0}^{\infty} q^{3n}q^{2n^2-n} = \sum_{n,r=0}^{\infty}\frac{q^{3(n+r)}q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_n(q^4;q^2)_{n+2r}}$$

Using [1, (1.1.7) p. 11] in the above equation we get,

$$(q^2;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2[(n+r)^2+(n+r)]}}{(q^2;q^2)_n(q^4;q^2)_{n+2r}} = \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \tag{3.5}$$

(vi) Substituting $\alpha_n = \dfrac{q^{3n}}{(q^2;q^2)_n}$ and $a = 1$ in (2.3),

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{3n}q^{2n^2-n}}{(q^2;q^2)_n} = \sum_{n,r=0}^{\infty}\frac{q^{3r}q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_r(q^2;q^2)_n(q;q^2)_{n+2r}}$$

Using [1, (10.1.2) p. 241] in the above equation we get,

$$(q;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2+2r-n}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r} = \frac{1}{(q^4,q^6;q^{10})_\infty} \tag{3.6}$$

(vii) Substituting $\alpha_n = \dfrac{q^{4n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n q^{2n^2-n}q^{4n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}} = \sum_{n,r=0}^{\infty}\frac{q^{4r}q^{n+r}q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_r(q^2;q^4)_{r+1}(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (59)] in above equation we get,

$$(q^2;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2+4r}}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^4)_{r+1}} = \frac{(q^2,q^{12},q^{14};q^{14})_\infty}{(q;q)_\infty} \tag{3.7}$$

(viii) Substituting $\alpha_n = \dfrac{q^{4n}(-q^2;q^4)_n}{(q^4;q^4)_n}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n q^{2n^2-n}q^{4n}(-q^2;q^4)_n}{(q^4;q^4)_n} = \sum_{n,r=0}^{\infty}\frac{q^{4r}(-q^2;q^4)_r q^{n+r}q^{2(n+r)^2-(n+r)}}{(q^4;q^4)_r(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (34)] in above equation we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2(n+r)^2+4r}(-q^2;q^4)_r}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^4;q^4)_r} = \frac{1}{(q^6,q^8,q^{10};q^{16})_\infty} \tag{3.8}$$

(ix) Substituting $\alpha_n = \dfrac{(-q^2;q^4)_n}{(q^4;q^4)_n}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n q^{2n^2-n}(-q^2;q^4)_n}{(q^4;q^4)_n} = \sum_{n,r=0}^{\infty} \frac{(-q^2;q^4)_r q^{n+r} q^{2(n+r)^2-(n+r)}}{(q^4;q^4)_r(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (36)] in the above equation we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2(n+r)^2}(-q^2;q^4)_r}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^4;q^4)_r} = \frac{1}{(q^2,q^8,q^{14};q^{16})_\infty} \tag{3.9}$$

(x) Substituting $\alpha_n = \dfrac{1}{(q^2;q^4)_n(q^2;q^2)_n}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n q^{2n^2-n}}{(q^2;q^4)_n(q^2;q^2)_n} = \sum_{n,r=0}^{\infty} \frac{q^{n+r} q^{2(n+r)^2-(n+r)}}{(q^2;q^4)_r(q^2;q^2)_r(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (61)] in the above equation we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2(n+r)^2}}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^2;q^4)_r(q^2;q^2)_r} = \frac{(q^{12},q^{16},q^{28};q^{28})_\infty}{(q^2;q^2)_\infty} \tag{3.10}$$

(xi) Substituting $\alpha_n = \dfrac{q^{2n}}{(q^2;q^4)_{n+1}(q^2;q^2)_n}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n q^{2n^2-n} q^{2n}}{(q^2;q^2)_n(q^2;q^4)_{n+1}} = \sum_{n,r=0}^{\infty} \frac{q^{2r} q^{n+r} q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_r(q^2;q^4)_{r+1}(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (60)] in the above equation we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2(n+r)^2+2r}}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^4)_{r+1}} = \frac{(q^8,q^{20},q^{28};q^{28})_\infty}{(q^2;q^2)_\infty} \tag{3.11}$$

(xii) Substituting $\alpha_n = \dfrac{q^{n^2}(q;q)_{3n}}{(q^3;q^3)_n(q^3;q^3)_{2n}}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n q^{2n^2-n}q^{n^2}(q;q)_{3n}}{(q^3;q^3)_n(q^3;q^3)_{2n}} = \sum_{n,r=0}^{\infty}\frac{q^{r^2}(q;q)_{3r}q^{n+r}q^{2(n+r)^2-(n+r)}}{(q^3;q^3)_r(q^3;q^3)_{2r}(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (42)] in the above equation we get,

$$(q^2;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2+r^2}(q;q)_{3r}}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^3;q^3)_r(q^3;q^3)_{2r}} = \frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{3.12}$$

(xiii) Substituting $\alpha_n = \dfrac{q^{2n}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{2n+1}}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n q^{2n^2-n}q^{2n}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{2n+1}} = \sum_{n,r=0}^{\infty}\frac{(q^6;q^6)_r q^{2r}q^{n+r}q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_r(q^2;q^2)_{2r+1}(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (92)] in the above equation we get,

$$(q^2;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2+2r}(q^6;q^6)_r}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^2)_{2r+1}} = \frac{(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty} \tag{3.13}$$

(xiv) Substituting $\alpha_n = \dfrac{q^{4n}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{2n+2}}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n q^{2n^2-n}q^{4n}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{2n+2}} = \sum_{n,r=0}^{\infty}\frac{(q^6;q^6)_r q^{4r}q^{n+r}q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_r(q^2;q^2)_{2r+2}(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (91)] in the above equation we get,

$$(q^2;q^2)_\infty\sum_{n,r=0}^{\infty}\frac{q^{2(n+r)^2+4r}(q^6;q^6)_r}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^2)_{2r+2}} = \frac{(q^{12},q^{42},q^{54};q^{54})_\infty}{(q^2;q^2)_\infty} \tag{3.14}$$

(xv) Substituting $\alpha_n = \dfrac{q^{6n}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{n+2}}$ and $a = q$ in (2.3),

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n q^{2n^2-n}q^{6n}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{n+2}} = \sum_{n,r=0}^{\infty}\frac{(q^6;q^6)_r q^{6r}q^{n+r}q^{2(n+r)^2-(n+r)}}{(q^2;q^2)_r(q^2;q^2)_{r+2}(q^2;q^2)_n(q^2;q^2)_{n+2r}}$$

Using [8, (90)] in the above equation we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2(n+r)^2+6r}(q^6;q^6)_r}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^2)_{r+2}} = \frac{(q^6,q^{48},q^{54};q^{54})_\infty}{(q^2;q^2)_\infty} \tag{3.15}$$

4. Theorem 4.1. *Let us assume* $u_r = \frac{\left(\frac{k^2}{a};q^2\right)_r}{(q^2;q^2)_r}$ *and* $v_r = \frac{(k^2;q^2)_r}{(aq;q^2)_r}$ *where* $k \neq 0$, *and if*

$$\beta_n = \sum_{r=0}^{n} \frac{\left(\frac{k^2}{a};q^2\right)_{n-r}(k^2;q^2)_{n+r}}{(q^2;q^2)_{n-r}(aq;q^2)_{n+r}} \alpha_r(a,k) \tag{4.1}$$

$$\gamma_n = \sum_{r=0}^{n} \frac{\left(\frac{k^2}{a};q^2\right)_{r}(k^2;q^2)_{r+2n}}{(q^2;q^2)_{r}(aq;q^2)_{r+2n}} \delta_{r+n}(a,k) \tag{4.2}$$

gives the following result, under suitable convergence conditions:

$$(aq;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{a^{n+2r}q^{2n^2+4r^2+4nr-(n+r)}}{(q^2;q^2)_n(aq;q^2)_{n+2r}} \alpha_r(a,k) = \sum_{n=0}^{\infty} a^{2n}q^{4n^2-n}\alpha_n(a,k) \tag{4.3}$$

Proof. On substituting the above value of β_n in earlier stated equation (1.9) and applying the identity (1.8) we get,

$$\sum_{n=0}^{\infty} \alpha_n\gamma_n = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{\left(\frac{k^2}{a};q^2\right)_n (k^2;q^2)_{n+2r}\alpha_r(a,k)\delta_{n+r}(a,k)}{(q^2;q^2)_n(aq;q^2)_{n+2r}} \tag{4.4}$$

Let us assume $\delta_r = \left(\frac{a^2q}{k^4}\right)^r$. Substituting the value of δ_r in equation (4.2) we get,

$$\gamma_n = \sum_{r=0}^{\infty} \frac{\left(\frac{k^2}{a};q^2\right)_{r}(k^2;q^2)_{r+2n}}{(q^2;q^2)_{r}(aq;q^2)_{r+2n}} \left(\frac{a^2q}{k^4}\right)^{n+r} \tag{4.5}$$

After further simplification the equation reduces to

$$\gamma_n = \frac{(k^2;q^2)_{2n}}{(aq;q^2)_{2n}} \left(\frac{a^2q}{k^4}\right)^n {}_2\Phi_1\left[\begin{matrix}\frac{k^2}{a}, k^2q^{4n} \\ aq^{1+4n}\end{matrix}; q^2, \frac{a^2q}{k^4}\right] \tag{4.6}$$

Using the identity ${}_2\Phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}$ we get

$$\gamma_n = \frac{\left(\frac{a^2q}{k^2}, \frac{aq}{k^2}; q^2\right)_\infty (k^2; q^2)_{2n}}{\left(\frac{a^2q}{k^4}, aq; q^2\right)_\infty \left(\frac{a^2q}{k^2}; q^2\right)_{2n}} \left(\frac{a^2q}{k^4}\right)^n \tag{4.7}$$

Substituting γ_n and δ_n in (4.4) we get,

$$\sum_{n=0}^{\infty} \frac{\left(\frac{a^2q}{k^2}, \frac{aq}{k^2}; q^2\right)_\infty (k^2; q^2)_{2n}}{\left(\frac{a^2q}{k^4}, aq; q^2\right)_\infty \left(\frac{a^2q}{k^2}; q^2\right)_{2n}} \left(\frac{a^2q}{k^4}\right)^n \alpha_n(a, k)$$
$$= \sum_{n,r=0}^{\infty} \frac{\left(\frac{k^2}{a}; q^2\right)_n (k^2; q^2)_{n+2r}}{\left(q^2; q^2\right)_n \left(aq; q^2\right)_{n+2r}} \left(\frac{a^2q}{k^4}\right)^{n+r} \alpha_r(a, k) \tag{4.8}$$

Provided all infinite series converges and $\alpha_0(a, k) = 1$.
As $k \to \infty$ the equation (4.8) yields,

$$(aq; q^2)_\infty \sum_{n,r=0}^{\infty} \frac{a^{n+2r} q^{2n^2+4r^2+4nr-(n+r)}}{(q^2; q^2)_n (aq; q^2)_{n+2r}} \alpha_r(a, k) = \sum_{n=0}^{\infty} a^{2n} q^{4n^2-n} \alpha_n(a, k),$$

which is precisely (4.3).

5. Special Cases of Theorem 4.1.

(i) Let $a = 1$ and $\alpha_n = \frac{q^{n-2n^2}}{(q^2; q^2)_n (-q; q)_{2n}}$ in (4.3) we get,

$$(q; q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+4r^2+4nr-(n+r)} q^{r-2r^2}}{(q^2; q^2)_n (q; q^2)_{n+2r} (q^2; q^2)_r (-q; q)_{2r}} = \sum_{n=0}^{\infty} \frac{q^{4n^2-n} q^{n-2n^2}}{(q^2; q^2)_n (-q; q)_{2n}}$$

Using identity [8, (33)] we get,

$$(q; q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+2r^2+4nr-n}}{(q^2; q^2)_n (q; q^2)_{n+2r} (q^2; q^2)_r (-q; q)_{2r}} = \frac{(q^3, q^4, q^7; q^7)_\infty}{(q^2; q^2)_\infty} \tag{5.1}$$

(ii) Let $a = q$ and $\alpha_n = \dfrac{q^{n-2n^2}}{(q^2;q^2)_n(-q;q)_{2n}}$ in (4.3) we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n+2r}q^{2n^2+4r^2+4nr-(n+r)}q^{r-2r^2}}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q^2;q^2)_r(-q;q)_{2r}} = \sum_{n=0}^{\infty} \frac{q^{2n}q^{4n^2-n}q^{n-2n^2}}{(q^2;q^2)_n(-q;q)_{2n}}$$

Using identity [8, (32)] we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+2r^2+4nr+2r}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r(-q;q)_{2r}} = \frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty} \tag{5.2}$$

(iii) Let $a = 1$ and $\alpha_n = \dfrac{q^{n-2n^2}}{(q^2;q^2)_n}$ in (4.3) we get,

$$(q;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+4r^2+4nr-(n+r)}q^{r-2r^2}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r} = \sum_{n=0}^{\infty} \frac{q^{4n^2-n}q^{n-2n^2}}{(q^2;q^2)_n}$$

Using equation [1, (10.1.1), p. 241] we get,

$$(q;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+2r^2+4nr-n}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r} = \frac{1}{(q^4,q^8;q^{10})_\infty} \tag{5.3}$$

(iv) Let $a = q$ and $\alpha_n = q^{n-2n^2}$ in (4.3) we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n+2r}q^{2n^2+4r^2+4nr-(n+r)}q^{r-2r^2}}{(q^2;q^2)_n(q;q^2)_{n+2r}} = \sum_{n=0}^{\infty} q^{2n}q^{4n^2-n}q^{n-2n^2}$$

Using equation [1, (1.1.1), p. 11] we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+2r^2+4nr+2r}}{(q^2;q^2)_n(q^2;q^2)_{n+2r}} = \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \tag{5.4}$$

(v) Let $a = 1$ and $\alpha_n = \dfrac{q^{3n-2n^2}}{(q^2;q^2)_n}$ in (4.3) we get,

$$(q;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+4r^2+4nr-(n+r)}q^{3r-2r^2}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r} = \sum_{n=0}^{\infty} \frac{q^{4n^2-n}q^{3n-2n^2}}{(q^2;q^2)_n}$$

Using equation [1, (10.1.2), p. 241] we get,

$$(q;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+2r^2+4nr+2r-n}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r} = \frac{1}{(q^4,q^6;q^{10})_\infty} \tag{5.5}$$

(vi) Let $a=1$ and $\alpha_n = \dfrac{q^{n-n^2}(q;q)_{3n}}{(q^3;q^3)_n(q^3;q^3)_{2n}}$ in (4.3) we get,

$$(q;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+4r^2+4nr-(n+r)}q^{r-3r^2}(q;q)_{3r}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^3;q^3)_r(q^3;q^3)_{2r}} = \sum_{n=0}^{\infty} \frac{q^{4n^2-n}q^{n-3n^2}(q;q)_{3n}}{(q^3;q^3)_n(q^3;q^3)_{2n}}$$

Using identity [8, (42)] we get,

$$(q;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+3r^2+4nr-n}(q;q)_{3r}}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^3;q^3)_r(q^3;q^3)_{2r}} = \frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{5.6}$$

(vii) Let $a=q$ and $\alpha_n = \dfrac{q^{2n-3n^2}(q^3;q^3)_n}{(q;q)_n(q;q)_{n+2}}$ in (4.3) we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n+2r}q^{2n^2+4r^2+4nr-(n+r)}q^{2r-3r^2}(q^3;q^3)_r}{(q^2;q^2)_n(q;q^2)_{n+2r}(q;q)_r(q;q)_{r+2}}$$
$$= \sum_{n=0}^{\infty} \frac{q^{2n}q^{4n^2-n}q^{2n-3n^2}(q^3;q^3)_n}{(q;q)_n(q;q)_{n+2}}$$

Using identity [8, (90)] we get,

$$(q^2;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+r^2+4nr+3r}(q^3;q^3)_r}{(q^2;q^2)_n(q^2;q^2)_{n+2r}(q;q)_r(q;q)_{r+2}} = \frac{(q^3,q^{24},q^{27};q^{27})_\infty}{(q;q)_\infty} \tag{5.7}$$

(viii) Assuming $a=1$ and $\alpha_n = \dfrac{q^{n-3n^2}(-q;q^2)_n}{(q^2;q^2)_n}$ in (4.3) we get,

$$(q;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+4r^2+4nr-(n+r)}q^{r-3r^2}(-q;q^2)_r}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r} = \sum_{n=0}^{\infty} \frac{q^{4n^2-n}q^{n-3n^2}(-q;q^2)_n}{(q^2;q^2)_n}$$

Using identity [8, (36)] we get,

$$(q;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+r^2+4nr-n}(-q;q^2)_r}{(q^2;q^2)_n(q;q^2)_{n+2r}(q^2;q^2)_r} = \frac{1}{(q,q^4,q^7;q^8)_\infty} \tag{5.8}$$

(ix) Taking $a = q^2$ and $\alpha_n = \dfrac{q^{n-2n^2}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{2n+2}}$ in (4.3) we get,

$$(q^3;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2(n+2r)}q^{2n^2+4r^2+4nr-(n+r)}q^{r-2r^2}(q^6;q^6)_r}{(q^2;q^2)_n(q^3;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^2)_{2r+2}}$$
$$= \sum_{n=0}^{\infty} \frac{q^{4n}q^{4n^2-n}q^{n-2n^2}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{2n+2}}$$

Using identity [8, (91)] we get,

$$(q^3;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+2r^2+4nr+n+4r}(q^6;q^6)_r}{(q^2;q^2)_n(q^3;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^2)_{2r+2}} = \frac{(q^{12},q^{42},q^{54};q^{54})_\infty}{(q^2;q^2)_\infty} \quad (5.9)$$

(x) Taking $a = q^2$ and $\alpha_n = \dfrac{q^{n-2n^2}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}$ in (4.3) we get,

$$(q^3;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2(n+2r)}q^{2n^2+4r^2+4nr-(n+r)}q^{r-2r^2}}{(q^2;q^2)_n(q^3;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^4)_{r+1}}$$
$$= \sum_{n=0}^{\infty} \frac{q^{4n}q^{4n^2-n}q^{n-2n^2}}{(q^2;q^2)_n(q^2;q^4)_{n+1}}$$

Using identity [8, (59)] we get,

$$(q^3;q^2)_\infty \sum_{n,r=0}^{\infty} \frac{q^{2n^2+2r^2+4nr+n+4r}}{(q^2;q^2)_n(q^3;q^2)_{n+2r}(q^2;q^2)_r(q^2;q^4)_{r+1}} = \frac{(q^4,q^{24},q^{28};q^{28})_\infty}{(q^2;q^2)_\infty} \quad (5.10)$$

Similarly various other double series identities can be derived.

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