South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.13, No.2, 2017, pp. 01-04

CONCERNING THE CONJUGATE OF A PARTITION

Neville Robbins

Mathematics Department
San Francisco State University San Francisco, CA 94132 USA
E-mail: nrobbins@sfsu.edu

(Received: July 21, 2017)

Abstract: We develop an algebraic formula for the conjugate of a partition. As an immediate consequence, we obtain an alternate proof for the known result that the number of distinct parts of a partition is invariant under conjugation. In addition, we present a theorem concerning the multiplicities of the parts of a partition.

Keywords and Phrases: Conjugate partition, distinct partition.

2010 Mathematics Subject Classification: 11P81.

1. Introduction

Let λ be a partition of the natural number n specified by

$$n = n_1 + n_2 + n_3 + \cdots + n_r \tag{1}$$

where the n_i are natural numbers such that

$$n_i \geq n_{i+1} \quad for\ all\ i. \tag{2}$$

Note that r represents the total number of parts in λ. The Ferrers graph of λ is a left-justified array consisting of n_i dots in the i^{th} row, where $1 \leq i \leq r$. This graph contains columns as well as rows. The *conjugate* of λ, denoted λ^*, is the partition obtained by interchanging the rows and columns of the Ferrers graph of λ. (This operation can also be called reflection about the main diagonal.) For an alternate definition of conjugate partition, see [4], Definition 1.8 on p.7 .

In this note, we obtain a formula for λ^*. We also show that the number of distinct parts of a partition is invariant under conjugation. This result has been previously stated, but not proven, by K. Alladi. (See [1], [2].) We also mention a simple proof recently offered by K. Alladi. (See [3].) Finally, we present a result concerning the multiplicities of the parts of a partition.

We use the notation that n^m represents m copies of n, and multiplication represents addition. Thus $3^2 2^3$ represents the partition 3+3+2+2+2.

It is sometimes useful to use the alternate notation:

$$n = \prod_{i=1}^{s} n_i^{a_i} \tag{3}$$

where the n_i are natural numbers such that

$$n_i > n_{i+1} \quad for\ all\ i \tag{4}$$

Remarks: Note that s represents the number of distinct parts in λ.

2. The Main Results

It is known that the largest part in λ is the number of parts in λ^*, and vice versa. We begin with a formula for λ^*.

Theorem 1. *Let λ be a partition of the natural number n, as specified by (3) and (4) above. Then the conjugate partition λ^* is given by:*

$$n = \prod_{j=1}^{s} m_j^{n_j} \tag{5}$$

where each

$$m_j = \prod_{i=1}^{j} a_i\ . \tag{6}$$

Proof. Each repeated part in λ (if any) corresponds bijectively to a missing part in λ^*. If we let s, s^* represent respectively the number of distinct parts in λ, λ^*, then we have: s = r - (number of repeated parts in λ) = r - (number of missing parts in λ^*) = s^*.

Remarks: We could also write the conclusion of Theorem 1 using the notation of (1) and (2), namely:

$$n = r^{n_r} \prod_{i=1}^{r-1} (r-i)^{n_{r-i}-n_{r-i+1}} = r^{n_r} \prod_{j=1}^{r-1} j^{n_j - n_{j+1}}\ . \tag{7}$$

Note that the part j occurs $n_j - n_{j+1}$ times; if $n_j - n_{j+1} = 0$, then j does not occur in λ^*. For example, we have

$$(522)^* = 3^2 2^{2-2} 1^{5-2} = 3^2 1^3 = 33111;$$

$$(33111)^* = 5^1 4^{1-1} 3^{1-1} 2^{3-1} 1^{3-3} = 5^1 2^2 = 522.$$

Also note that Theorem 1 is equivalent to Definition 1.8 on p.7 of [4].

Corollary. *The number of distinct parts of a partition is invariant under conjugation.*

Proof. This follows from Theorem 1, since s is the number of distinct parts in both λ and λ^*.

Alternate Proof of Corollary. In the Ferrers graph of a partition, call a node a *corner* if (i) it is the rightmost node in its row, and (ii) there are no nodes directly below it. Clearly, the number of corners is the same as the number of distinct parts of the partition. Also, corners are preserved by conjugation. The conclusion now follows.

Remarks: This alternate proof is due to K. Alladi [3].

Definition 1. *A partition is said to have degree d if d is the least positive integer such that each part occurs at least d times.*

Theorem 2. *For any natural number n, the partitions of n with degree at least d are equinumerous with the partitions of n such that (i) the least part is at least d, and (ii) the difference between any two distinct consecutive parts is at least d.*

Proof. If λ is a partition of n with degree d, then by Theorem 1, the repeated parts in λ correspond bijectively to the missing parts in λ^*. The conclusion now follows.

Remarks: In [5], we referred to partitions of degree at least 2 as *nosolo* partitions. Theorem 2 generalizes a theorem attributed by Andrews to MacMahon. (See [4] , 9. on p.14.)

References

[1] K. Alladi, A fundamental invariant in the theory of partitions, Proceedings, 1997 Conference at Pennsylvania State University 101-113

[2] K. Alladi, A fundamental but unexploited partition invariant, Proceedings, Kyoto Conference on Number Theory Kluwer (1999) 19-23

[3] K. Alladi, (private communication)

[4] George E. Andrews, *The Theory of Partitions* (1984) Cambridge University Press

[5] N. Robbins, On the number of nosolo partitions of n *Pioneer J. Math. & Math. Sci.* **17** (2016) 33-39