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DECIMAL EXPANSIONS: THEIR UNIQUENESS AND INTERESTING PATTERNS

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Dedicated to Prof. K. Srinivasa Rao on his 75^{th} ***Birth Anniversary***

Abstract: An expression of the form o.abcd... written by arbitrarily picking up a, b, c, d ... from among 0 to 9, cannot be taken to represent the decimal expansion of a real number. In fact, the decimal expansion of every irrational number is unique in itself whereas those of rational numbers (fractions) follow some interesting patterns. These results are of far-reaching consequences in the context of Cantor's second proof (1878) about the uncountability of real numbers.

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1. Introduction

Representation of rational numbers (fractions) and irrational numbers in decimal system facilitate in arithmetical operations. It is well-known that decimal expansion of every irrational number is unique in itself [1]. But, herein it has been shown that the decimal expansions of rational numbers (fractions) have some very interesting patterns. These results are of far reaching consequences in the context of Cantor's second proof (1878) about the uncountability of real numbers [2].

2. Some interesting patterns in the decimal expansions of fractions

Fractions of the form a/b, subject to $1 \leq a \leq b-1$ have either terminating decimal expansions (tde) or non-tde. In tde, the digits to the right of the decimal point are finite e.g. $\frac{1}{2} = 0.5$. But, in non-tde, the digits to the right of the decimal point are infinite but repeating themselves after finite number of digits, called its period P e.g. $1/3 = 0.333...$, $1/11 = 0.0909....$, $1/37 = 0.027027....$ have $P = 1, 2, 3$ respectively. In general, if p is a prime then P of $1/p$ is given by $P = (p-1)/j$

for some $j \geq 1$; in the above three cases $p = 3, 11, 37$, $P = 1, 2, 3$, and $j = 2, 5, 12$ respectively.

It will now be shown that the non-tdes of the fractions of the form a/p for $2 \leq a \leq p-1$, have particular patterns originating from the corresponding unit fraction $1/p$. In this context, note that non-tdes with period P can also be expressed in the form of infinite series $\sum_{k=1}^{\infty} a_k/10^k$, where $a_k s$ repeat their values after each P. Now proceed, as follows.

3. First interesting pattern

To discover first hidden pattern, express the non-tdes of $1/7$ to $6/7$, as follows. $1/7 = 0.142857.... = \sum a_k/10^k$, say, so that $a_1 = 1$, $a_2 = 4$, $a_3 = 2$, $a_4 = 8$, $a_5 = 5$, $a_6 = 7$ and then a_7 to a_{12} onwards will repeat the values of a_1 to a_6 respectively; here note that $a_1 < a_3 < a_2 < a_5 < a_6 < a_4$ for future reference. Now, note that non-tdes of $2/7$ to $6/7$ can also be expressed in terms of $a_k s$ as given by

$2/7 = 0.285714.... = 0.a_3a_4a_5a_6a_1a_2...$

$3/7 = 0.428571.... = 0.a_2a_3a_4a_5a_6a_1...$

$4/7 = 0.571428.... = 0.a_5a_6a_1a_2a_3a_4...$

$5/7 = 0.714285.... = 0.a_6a_1a_2a_3a_4a_5....$

$6/7 = 0.857142.... = 0.a_4a_5a_6a_1a_2a_3....$

From the forms of above non-tdes, the pattern discovered can be described, as follows.

(i) The first digits of $2/7$ to $6/7$ viz. a_3, a_2, a_5, a_6, a_4 are in ascending order.

(ii) The corresponding remaining digits of $2/7$ to $6/7$ rotate about these first digits a_3, a_2, a_5, a_6, a_4.

4. Second Interesting pattern

In the above example of $a/7$, $1 \leq a \leq 6$, $P = 6$ is maximum $(a = 6)$ and hence in such cases similar pattern will follow e.g. in $a/17$, $a \leq 1 \leq 16$ or $a/19$, $a \leq 1 \leq 18$, $P = 16$ or 18 is maximum $(a = 16$ or $18)$. But, in other cases where P is not maximum, there will be a systematic change in deciding the first digits of the non-tdes of a/p, $2 \leq a \leq p-1$, from that of $1/p$; this is explained below by taking another example.

Express the non-tdes of $1/13$ and $2/13$ as follows.
$1/13 = 0.076923... = \sum a_k/10^k$, say, so that $a_1 = 0, a_2 = 7, a_3 = 6, a_4 = 9, a_5 = 2, a_6 = 3$.

$2/13 = 0.153846.... = \sum b_k/10^k$, say, so that $b_1 = 1, b_2 = 5, b_3 = 3, b_4 = 8, b_5 = 4, b_6 = 6$.

Now, for expressing the non-tdes of 3/13 to 12/13, note that as in $a/7$ above, here all $a_k s$ do not repeat in $b_k s$ except $a_3 = b_6 = 6$ and $a_6 = b_3 = 3$. Hence, arrange $a_k s$ and $b_k s$ in ascending order, as follows.

$$a_1 < b_1 < a_5 < a_6, b_3 < b_5 < b_2 < a_3, b_6 < a_2 < b_4 < a_4$$

Now, for deciding the order of ascendance between equal value of $a_6, b_3 (= 3)$ and $a_3, b_6 (= 6)$, consider them along with their corresponding next digits. Thus, a_6 and b_3, and, a_3 and b_6, are to be viewed as $a_6 a_1 = 30$ and $b_3 b_4 = 38$, and, $a_3 a_4 = 69$ and $b_6 b_1 = 61$. Accordingly, deduce that $a_6 < b_3$ and $b_6 < a_3$, and, the above ascending order among $a_k s$ and $b_k s$ can be finally arranged, as follows.

$$a_1 < b_1 < a_5 < a_6 < b_3 < b_5 < b_2 < b_6 < a_3 < a_2 < b_4 < a_4.$$

Now, non-tdes of 3/13 to 12/13 can be expressed by adopting the pattern described in (ii) of sec. 3 without actually performing the process of division; these given below.

$3/13 = 0.a_5 a_6 a_1 a_2 a_3 a_4... = 0.230769...$

$4/13 = 0.a_6 a_1 a_2 a_3 a_4 a_5... = 0.307692...$

$5/13 = 0.b_3 b_4 b_5 b_6 b_1 b_2... = 0.384615...$

$6/13 = 0.b_5 b_6 b_1 b_2 b_3 b_4... = 0.461538...$

$7/13 = 0.b_2 b_3 b_4 b_5 b_6 b_1... = 0.538461...$

$8/13 = 0.b_6 b_1 b_2 b_3 b_4 b_5... = 0.615384...$

$9/13 = 0.a_3 a_4 a_5 a_6 a_1 a_2... = 0.692307...$

$10/13 = 0.a_2 a_3 a_4 a_5 a_6 a_1... = 0.769230...$

$11/13 = 0.b_4 b_5 b_6 b_1 b_2 b_3... = 0.846153...$

$12/13 = 0.a_4 a_5 a_6 a_1 a_2 a_3... = 0.923076...$

After studying the non-tdes of some other fractions,following additional patterns have been discovered.

5. Third Interesting pattern

(a) The non-tdes of 1/11 to 10/11 can be expressed by $0.(a)(9 - a)$ where 'a' varies from 0 to 9 giving $1/11 = 0.09...$, $2/11 = 0.18....$, $3/11 = 0.27.....$, $10/11 = 0.90$.

(b) The non-tdes of 1/101 to 100/101 can be expressed by $0.(a)(99-a)$ where $a = 00$ to 99 giving $1/101 = 0.0099...$, $2/101 = 0.0198....$, $3/101 = 0.0297.....$, $100/101 = 0.9900....$

6. Deep-rooted patterns

Lastly, following interesting features will further reveal the deep-rooted patterns existing among the non-tdes of $1/p$ where p is a prime.

(i) When $p = 11, 31, 41, 101, 271, ...$ with last digit 1, then the last digit before repetition of the non-tdes of $1/p$ will be 9, as can be confirmed from (a) and (b) above; refer to $1/11 = 0.09...$ and $1/101 = 0.0099....$

(ii) On the other hand, when $p = 19, 29, 239, ...$ with last digit 9 then the last digit before repetition of the non-tdes of $1/p$ will be 1 e.g. $1/19 = 0.05263157$ 89473 68421....

(iii) When $p = 3, 13, 23, ...$ with last digit 3, then the last digit before repetition of the non-tdes of $1/p$ will also be 3, as can be confirmed from $1/3 = 0.3...$ or $1/13 = 0.76923...$ and $2/13 = 0.153846...$ wherein last digits are 3 and 6 (2 times 3); refer to sec. 4.

(iv) When $p = 7, 17, 37, 137, ...$ with last digit 7, then also the last digit before repetition of the non-tdes of $1/p$ will be 7 as can be confirmed from $1/7 = 0.142857...$, refer to sec. 3.

Regarding non-repeating decimal expansions of irrational numbers, each one of them is unique in itself; this can be confirmed by taking individual cases such as $\sqrt{2} = 1.41421...$, $\sqrt{3} = 1.73205...$ etc.

7. Remark

Cantor, in his second proof for the uncountability of real numbers has created a real number by arbitrarily picking up digits from the assumed one-to-one mapping of the real numbers [2].

References

[1] Clawson Calvin C., The Mathematical Traveler : Exploring the Grand History of Numbers, Viva Books Pvt. Ltd., India, (2004); Chaps. 9; 173-178.

[2] Ibid., Chap. 10; 201-202.

[3] Gurtu, Vishnu K., Exploring Infinity Mathematically, South East Asian J. of Math. & Math. Sci. Vol.13, No.1 2017, pp. 87-92.