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ON CERTAIN TRANSFORMATIONS OF BASIC HYPERGEOMETRIC FUNCTIONS USING BAILEY'S TRANSFORM

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Dedicated to Prof. K. Srinivasa Rao on his 75^{th} Birth Anniversary

Abstract: In this paper, making use of Bailey transform and certain known summation formulas, we have established certain interesting transformation formulas of basic hypergeometric series.

Keywords and Phrases: Basic hypergeometric series, Bailey's pair and Bailey's transformation.

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1. Introduction

The generalized basic hyper geometric series ${}_r\phi_s$ is defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_n)_n}{(q, b_1, b_2, \ldots, b_n)_n} [(-1)^n q^{n(n-1)/2}]^{1+s-r} z^n \quad (1.1) $$

where r and s are positive integers and $|q| < 1$. The above series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s + 1$.
For real or complex a, $q < 1$, the q -shifted factorial is defined by

$$ (a, q)_n = \begin{cases} 1 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2)\ldots,(1-aq^{n-1}) & \text{if } n \in N. \end{cases} \quad (1.2) $$

In 1947, Bailey established a remarkably simple and useful transformation formula which is given in the following form:
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.3}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \tag{1.4}$$

where α_r,δ_r,u_r and v_r are functions of r only such that the series of γ_n exists, then under suitable conditions of convergence we have the following equation.

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.5}$$

In this paper, we shall use the following results due to Verma and Jain [10].

$$ {}_4\Phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; -q^{-1/2+n} \\ \sqrt{a}, -\sqrt{a}, aq^{1+n} \end{matrix} \right] $$

$$ = \frac{(aq;q)_n}{2} \left[\frac{(-q^{-1/2};q)_n}{(\sqrt{aq};q)_n(-q\sqrt{a};q)_{n-1}} + \frac{(-q^{-1/2};q)_n}{(-\sqrt{aq};q)_n(q\sqrt{a};q)_{n-1}} \right] \tag{1.6}$$

$$ {}_3\Phi_2 \left[\begin{matrix} a, q\sqrt{a}, q^{-n}; -q^{n} \\ \sqrt{a}, aq^{1+n} \end{matrix} \right] = \frac{(aq,-1;q)_n}{2} \left[\frac{(1+\sqrt{a})}{(aq;q^2)_n} + \frac{(1-\sqrt{a})}{(\sqrt{a};q)_n(-q\sqrt{a};q)_n} \right] \tag{1.7}$$

$$ {}_2\Phi_1 \left[\begin{matrix} a, q^{-n}; -q^{1/2+n} \\ aq^{1+n} \end{matrix} \right] $$

$$ = \frac{(aq,-\sqrt{q};q)_n}{2} \left[\frac{(1+\sqrt{a})}{(-\sqrt{aq};q)_n(q\sqrt{a};q)_n} + \frac{(1-\sqrt{a})}{(\sqrt{aq};q)_n(-q\sqrt{a};q)_n} \right] \tag{1.8}$$

2. Main Results

In this section we shall establish the following results.

$$ {}_5\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c; q; \frac{-a\sqrt{q}}{bc} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, 0 \end{matrix} \right] = \frac{1}{2} \prod \left[\begin{matrix} aq, aq/bc; q \\ aq/b, aq/c \end{matrix} \right] \times $$

$$\left\{ {}_3\Phi_2 \left[\begin{array}{c} b, c, -\sqrt{a}; q; aq/bc \\ \\ \sqrt{aq}, -q\sqrt{a} \end{array} \right] + \sqrt{a}\ {}_3\Phi_2 \left[\begin{array}{c} b, c, -\sqrt{a}; q; a^{3/2}q^2/bc \\ \\ \sqrt{aq}, -q\sqrt{a} \end{array} \right] \right.$$

$$\left. + {}_3\Phi_2 \left[\begin{array}{c} b, c, -\sqrt{q}; q; aq/bc \\ \\ -\sqrt{aq}, q\sqrt{a} \end{array} \right] - \sqrt{a}{}_3\Phi_2 \left[\begin{array}{c} b, c, -\sqrt{q}; q; a^{3/2}q^2/bc \\ \\ -\sqrt{aq}, q\sqrt{a} \end{array} \right] \right\} \tag{2.1}$$

$${}_4\Phi_4 \left[\begin{array}{c} a, q\sqrt{a}, b, c; q; -aq/bc \\ \sqrt{a}, aq/b, aq/c, 0 \end{array} \right] = \frac{1}{2} \prod \left[\begin{array}{c} aq, aq/bc; q \\ aq/b, aq/c \end{array} \right] \times$$

$$\left\{ (1+\sqrt{a}){}_3\Phi_2 \left[\begin{array}{c} b, c, -1; q; \frac{aq}{bc} \\ \\ \sqrt{aq}, -\sqrt{aq} \end{array} \right] + (1-\sqrt{a})\ {}_3\Phi_2 \left[\begin{array}{c} b, c, -1; q; \frac{aq}{bc} \\ \\ \sqrt{a}, -q\sqrt{a} \end{array} \right] \right\} \tag{2.2}$$

$${}_3\Phi_3 \left[\begin{array}{c} a, b, c; q; \frac{-aq^{3/2}}{bc} \\ \\ aq/b, aq/c, 0 \end{array} \right] = \frac{1}{2} \prod \left[\begin{array}{c} aq, aq/bc; q \\ \\ aq/b, aq/c \end{array} \right] \times$$

$$\left\{ {}_3\Phi_2 \left[\begin{array}{c} b, c, -\sqrt{q}; q; aq/bc \\ \\ -\sqrt{aq}, \sqrt{aq} \end{array} \right] + \ {}_3\Phi_2 \left[\begin{array}{c} b, c, -\sqrt{q}; q; aq/bc \\ \\ -\sqrt{a}, q\sqrt{a} \end{array} \right] \right\} \tag{2.3}$$

$${}_7\Phi_7 \left[\begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e; q; \frac{a^2q^2}{bcde} \\ \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, aq/d, aq/e, 0 \end{array} \right]$$

$$= \frac{(aq, aq/de; q)_\infty}{(aq/d, aq/e; q)_\infty} {}_3\phi_2 \left[\begin{array}{c} a, e, aq/bc; q; \frac{aq}{de} \\ aq/b, aq/c \end{array} \right] \tag{2.4}$$

Proof of (2.1):

Let us choose $u_r = \dfrac{1}{(q;q)_r}, v_r = \dfrac{1}{(aq;q)_r},\ \alpha_r = \dfrac{(a, q\sqrt{a}, -q\sqrt{a}; q)_r q^{r(r-1)/2}}{(q, \sqrt{a}, -\sqrt{a}; q)_r q^{r/2}}$,
and $\delta_r = (b, c; q)_r (\frac{aq}{bc})^r$
Now using these in equations (1.2), (1.3) and using (1.6) we get the following:

$$\beta_n = \frac{(-q^{-1/2}; q)_n}{2} \left\{ \frac{1 + q^n\sqrt{a}}{(q, \sqrt{aq}, -q\sqrt{a}; q)_n} + \frac{1 - q^n\sqrt{a}}{(q, -\sqrt{aq}, q\sqrt{a}; q)_n} \right\} \tag{2.5}$$

and

$$\gamma_n = \frac{(b.c; q)_n}{(aq; q)_n} \left(\frac{aq}{bc} \right)^n {}_2\phi_1 \left[\begin{array}{c} bq^n, cq^n; q; \frac{aq}{bc} \\ aq^{1+2n} \end{array} \right] \tag{2.6}$$

Summing the series ${}_2\phi_1$, we have:

$$\gamma_n = \frac{(b.c;q)_n}{(aq/b,aq/c;q)_n}\left(\frac{aq}{bc}\right)^n \prod\left[\begin{matrix} aq/b, aq/c; q \\ aq, aq/bc \end{matrix}\right] \tag{2.7}$$

Putting these values in equation (1.5), we get the proof of (2.1).

Proof of (2.2):

Let us choose $u_r = \dfrac{1}{(q;q)_r}, v_r = \dfrac{1}{(aq;q)_r},\ \alpha_r = \dfrac{(a,q\sqrt{a};q)_r q^{r(r-1)/2}}{(q,\sqrt{a};q)_r}$,
and $\delta_r = (b,c;q)_r(\frac{aq}{bc})^r$
Now using these in equations (1.2), (1.3) and using (1.7) we get the following:

$$\beta_n = \frac{1}{(q,aq;q)_n}{}_3\Phi_2\left[\begin{matrix} a, q\sqrt{a}, q^{-n}; -q^n \\ \sqrt{a}, aq^{1+n} \end{matrix}\right]$$

$$= \frac{(-1;q)_n}{2(q;q)_n}\left\{\frac{1+\sqrt{a}}{(aq;q^2)_n} + \frac{1-\sqrt{a}}{(\sqrt{a},-q\sqrt{a};q)_n}\right\} \tag{2.8}$$

and

$$\gamma_n = \frac{(b,c;q)_n}{(aq;q)_n}\left(\frac{aq}{bc}\right)^n {}_2\phi_1\left[\begin{matrix} bq^n, cq^n; q; \frac{aq}{bc} \\ aq^{1+2n} \end{matrix}\right] \tag{2.9}$$

Summing the series ${}_2\phi_1$,we have:

$$\gamma_n = \frac{(b,c;q)_n}{(aq/b,aq/c;q)_n}\left(\frac{aq}{bc}\right)^n \prod\left[\begin{matrix} aq/b, aq/c; q \\ aq, aq/bc \end{matrix}\right] \tag{2.10}$$

Putting these values in equation (1.5), we get the proof of (2.2)

Proof of (2.3):

Let us choose $u_r = \dfrac{1}{(q;q)_r}, v_r = \dfrac{1}{(aq;q)_r},\ \alpha_r = \dfrac{(a;q)_r q^{r^2/2}}{(q;q)_r}$, and $\delta_r = (b,c;q)_r(\frac{aq}{bc})^r$
Now using these in equations (1.3), (1.4) and using (1.8) we get the following:

$$\beta_n = \frac{(-\sqrt{q};q)_n}{2(q;q)_n}\left\{\frac{1+\sqrt{a}}{(-\sqrt{aq},q\sqrt{a};q)_n} + \frac{1-\sqrt{a}}{(\sqrt{aq},-q\sqrt{a};q)_n}\right\} \tag{2.11}$$

and

$$\gamma_n = \frac{(b,c;q)_n}{(aq;q)_{2n}}\left(\frac{aq}{bc}\right)^n {}_2\phi_1\left[\begin{matrix} bq^n, cq^n; q; \frac{aq}{bc} \\ aq^{1+2n} \end{matrix}\right] \tag{2.12}$$

Summing the series ${}_2\phi_1$,we have:

$$\gamma_n = \frac{(b,c;q)_n}{(aq/b,aq/c;q)_n}\left(\frac{aq}{bc}\right)^n \prod\left[\begin{array}{l} aq/b, aq/c; q \\ aq, aq/bc \end{array}\right] \tag{2.13}$$

Putting these in equation (1.5), we get the proof of (2.3)
Proof of (2.4):
In order to prove (2.4) we shall use the following summation formulas:

$${}_6\phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n}; q; \frac{aq^{1+n}}{bc} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, aq^{1+n} \end{array}\right] = \frac{(aq,aq/bc;q)_n}{(aq/b,aq/c;q)_n} \tag{2.14}$$

$${}_2\phi_1\left[\begin{array}{l} a, b; q; \frac{c}{ab} \\ c \end{array}\right] = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty} \tag{2.15}$$

Let us choose
$u_r = \dfrac{1}{(q;q)_r}, v_r = \dfrac{1}{(aq;q)_r},\ \alpha_r = \dfrac{(a,q\sqrt{a},-q\sqrt{a},b,c;q)_r q^{r(r+1)/2}}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c;q)_r}\left(\dfrac{-a}{bc}\right)^r,$
and $\delta_r = (d,e;q)_r(\frac{aq}{de})^r$
Now using these in equations (1.3) and (1.4) we get the following:

$$\beta_n = \frac{1}{(q,aq;q)_n}{}_6\phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n}; q; \frac{-aq^{1+n}}{bc} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, aq^{1+n} \end{array}\right] \tag{2.16}$$

Now summing the series ${}_6\phi_5$ using equation (2.14), we get:

$$\beta_n = \frac{(aq/bc;q)_n}{(q,aq/b,aq/c;q)_n} \tag{2.17}$$

and

$$\gamma_n = \frac{(d,e;q)_n}{(aq;q)_{2n}}\left(\frac{aq}{de}\right)^n {}_2\phi_1\left[\begin{array}{l} dq^n, eq^n; q; \frac{aq}{de} \\ aq^{1+2n} \end{array}\right] \tag{2.18}$$

Summing the series ${}_2\phi_1$ using equation (2.15) we have:

$$\gamma_n = \frac{(d,e;q)_n}{(aq/d,aq/e;q)_n}\ \frac{(aq/d,aq/e;q)_\infty}{(aq,aq/de;q)_\infty}\left(\frac{aq}{de}\right)^n \tag{2.19}$$

Putting these values in equation (1.5), we get the proof of (2.4)

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