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ON CONTINUED FRACTIONS AND LAMBERT SERIES

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Dedicated to Prof. K. Srinivasa Rao on his 75^{th} Birth Anniversary

Abstract: In this paper, we have established certain results involving continued fractions and Lambert series.

Keywords and Phrases: Lambert series, continued fractions, q-shifted factorial, basic hypergeometric series.

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1. Introduction, Notations and Definitions

The q-shifted factorial is defined by,

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)...(1-aq^{n-1}), & n \geq 1 \end{cases}$$

Also,

$$(a;q)_{-n} = \frac{q^{n(n+1)/2}}{(-a)^n (q/a;q)_n}$$

The generalized basic hypergeometric series is given by,

$${}_r\Phi_{r-1}\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_{r-1} \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n ... (a_r;q)_n z^n}{(b_1;q)_n (b_2;q)_n ... (b_{r-1};q)_n (q;q)_n},$$

where max. $(|q|, |z|) < 1$.
A generalized bilateral basic hypergeometric series is defined by,

$${}_r\Psi_r\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_r \end{array}\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n (a_2;q)_n ... (a_r;q)_n z^n}{(b_1;q)_n (b_2;q)_n ... (b_r;q)_n},$$

where $\left|\frac{b_1, b_2, ..., b_r}{a_1, a_2, ..., a_r}\right| < |z| < 1$ for the convergence of the series.
An infinite continued fraction has the form,

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + ...}}}$$

To save space, this continued fraction is sometimes written as,

$$b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3 + ...}$$

There are a number of fascinating results on continued fractions in Notebooks of Ramanujan specially in the second and 'Lost' Notebooks. Many mathematicians such as B. Gordon [7], L. Carlitz [4], Hirschhorn [9], K.G. Ramanathan [13], G.E. Andrews & B.C. Berndt [1], S. Bhargava and C. Adiga [2], D.P. Gupta and D. Masson [8], R.Y. Denis [5,6], S.N. Singh [14], N. Bhagirathi [3], H.M. Srivastava et. al. [16] and many others have proved and also generalized the results of Ramanujan on continued fractions.

In this paper, we have made an attempt to give some interesting and new results on continued fractions.

2. Main Results

In this section, we shall establish our main results on continued fractions.

(i)

$$\sum_{n=0}^{\infty} \frac{\lambda^n q^{in}}{1-\mu q^{kn+j}} = \sum_{n=0}^{\infty} \frac{q^{kn^2+n(i+j)}(\lambda\mu)^n(1-\lambda\mu q^{2kn+i+j})}{(1-\mu q^{kn+j})(1-\lambda q^{kn+i})}$$

$$= \frac{1}{(1-\mu q^j)-}\frac{\lambda q^i(1-\mu q^j)^2}{(1-\mu q^{k+j})-}\frac{\lambda\mu q^{i+j}(1-q^k)^2}{(1-\mu q^{2k+j})-}\frac{\lambda q^{k+i}(1-\mu q^{k+j})^2}{(1-\mu q^{3k+j})-}$$

$$\frac{\lambda\mu q^{k+ij}(1-q^{2k})^2}{(1-\mu q^{4k+j})-}\frac{\lambda q^{2k+i}(1-\mu q^{2k+j})^2}{(1-\mu q^{5k+j})-}\frac{\lambda\mu q^{2k+i+j}(1-q^{3k})^2}{(1-\mu q^{6k+j})-}... \quad (2.1)$$

which is the continued fraction representation of the most general Lambert series.

(ii)

$$\sum_{n=-\infty}^{\infty} \frac{\lambda^n q^{in}}{1-\mu q^{kn+j}} = \sum_{n=0}^{\infty} \frac{(\lambda\mu)^n q^{kn^2+n(i+j)}(1-\lambda\mu q^{2kn+i+j})}{(1-\mu q^{kn+j})(1-\lambda q^{kn+i})}$$

$$-\frac{1}{\lambda\mu}\sum_{n=0}^{\infty}\frac{q^{kn^2+n(2k-i-j)+(k-i-j)}\left(1-\dfrac{q^{2kn+2k-i-j}}{\lambda\mu}\right)}{\left(1-\dfrac{q^{kn+k-i}}{\lambda}\right)\left(1-\dfrac{q^{kn+k-j}}{\mu}\right)(\lambda\mu)^n}$$

$$=\frac{(q^k;q^k)_\infty^2(\lambda\mu q^{i+j},q^{k-i-j}/\lambda\mu;q^k)_\infty}{(\mu q^j,q^{k-j}/\mu,\lambda q^i,q^{k-i}/\lambda;q^k)_\infty} \tag{2.2}$$

Proof of (2.1).

In order to prove (2.1) let us consider the ratio,

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(aq;q)_n(b;q)_nz^n}{(q;q)_n(c;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_nz^n}{(q;q)_n(c;q)_n}}=\cfrac{1}{1-\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(b;q)_nz^n}{(q;q)_n(c;q)_n}\{(aq;q)_n-(a;q)_n\}}{\displaystyle\sum_{n=0}^{\infty}\frac{(aq;q)_n(b;q)_nz^n}{(q;q)_n(c;q)_n}}}$$

$$=\cfrac{1}{1-\cfrac{az(1-b)/(1-c)}{1-\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(aq;q)_nz^n}{(q;q)_n}\left\{\frac{(bq;q)_n}{(cq;q)_n}-\frac{(b;q)_n}{(c;q)_n}\right\}}{\displaystyle\sum_{n=0}^{\infty}\frac{(aq;q)_n(bq;q)_nz^n}{(q;q)_n(cq;q)_n}}}}$$

$$=\cfrac{1}{1-\cfrac{az(1-b)/(1-c)}{1-\cfrac{z(b-c)(1-aq)/(1-c)(1-cq)}{1-\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(bq;q)_nz^n}{(q;q)_n}\left\{\frac{(aq^2;q)_n}{(cq^2;q)_n}-\frac{(aq;q)_n}{(cq;q)_n}\right\}}{\displaystyle\sum_{n=0}^{\infty}\frac{(aq^2;q)_n(bq;q)_nz^n}{(q;q)_n(cq^2;q)_n}}}}}$$

$$= \cfrac{1}{1 - \cfrac{az(1-b)/(1-c)}{1 - \cfrac{z(b-c)(1-aq)/(1-c)(1-cq)}{1 - \cfrac{zq(a-c)(1-bq)/(1-cq)(1-cq^2)}{1 - \cfrac{\sum_{n=0}^{\infty} \frac{(aq^2;q)_n z^n}{(q;q)_n} \left\{ \frac{(bq^2;q)_n}{(cq^3;q)_n} - \frac{(bq;q)_n}{(cq^2;q)_n} \right\}}{\sum_{n=0}^{\infty} \frac{(aq^2;q)_n (bq^2;q)_n z^n}{(q;q)_n (cq^3;q)_n}}}}}}$$

Iterating this process we get,

$$\frac{\sum_{n=0}^{\infty} \frac{(aq;q)_n (b;q)_n z^n}{(q;q)_n (c;q)_n}}{\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(q;q)_n (c;q)_n}} = \frac{1}{1-} \frac{az(1-b)/(1-c)}{1-} \frac{z(b-c)(1-aq)/(1-c)(1-cq)}{1-}$$

$$\frac{zq(a-c)(1-bq)/(1-cq)(1-cq^2)}{1-} \frac{zq(b-cq)(1-aq^2)/(1-cq^2)(1-cq^3)}{1-}$$

$$\frac{zq^2(a-cq)(1-bq^2)/(1-cq^3)(1-cq^4)}{1-\ldots} \tag{2.3}$$

Taking $a = 1$, using Rogers-Fine identity [1; (9.1.1), p. 223] and lastly applying [10; (2.3.14), p. 33] in (2.3) we get,

$$\sum_{n=0}^{\infty} \frac{(b;q)_n z^n}{(c;q)_n} = \sum_{n=0}^{\infty} \frac{(b;q)_n (bzq/c;q)_n c^n z^n q^{n^2-n} (1-bzq^{2n})}{(c;q)_n (z;q)_{n+1}}$$

$$= \frac{1}{1-} \frac{z(1-b)}{(1-c)-} \frac{z(b-c)(1-q)}{(1-cq)-} \frac{zq(1-c)(1-bq)}{(1-cq^2)-} \frac{zq(b-cq)(1-q^2)}{(1-cq^3)-}$$

$$\frac{zq^2(1-cq)(1-bq^2)}{(1-cq^4)-} \frac{zq^2(b-cq^2)(1-q^3)}{(1-cq^5)-\ldots} \tag{2.4}$$

Again, taking $c = bq$ in (2.4) we have,

$$\sum_{n=0}^{\infty} \frac{z^n}{1-bq^n} = \sum_{n=0}^{\infty} \frac{b^n z^n q^{n^2} (1-bzq^{2n})}{(1-bq^n)(1-zq^n)}$$

$$= \frac{1}{(1-b)-} \frac{z(1-b)^2}{(1-bq)-} \frac{bz(1-q)^2}{(1-bq^2)-} \frac{zq(1-bq)^2}{(1-bq^3)-} \frac{bzq(1-q^2)^2}{(1-bq^4)-}$$

$$\frac{zq^2(1-bq^2)^2}{(1-bq^5)-}\,\frac{bzq^2(1-q^3)^2}{(1-bq^6)-\ldots} \tag{2.5}$$

Now, first replacing q by q^k and then replacing z by λq^i and b by μq^j in (2.5) we get (2.1).

Proof of (2.2).

In order to prove (2.2) let us consider the Ramanujan's ${}_1\Psi_1$ summation formula, viz.,

$${}_1\Psi_1\left[\begin{matrix} a;q;z \\ b \end{matrix}\right] = \sum_{n=-\infty}^{\infty}\frac{(a;q)_n z^n}{(b;q)_n} = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty} \tag{2.6}$$

[15; App IV (IV. 12)]

which can be written as,

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n z^n}{(b;q)_n} = \sum_{n=0}^{\infty}\frac{(a;q)_n z^n}{(b;q)_n} + \frac{(1-\frac{q}{b})}{(1-\frac{q}{a})}\left(\frac{b}{az}\right)\sum_{n=0}^{\infty}\frac{(\frac{q^2}{b};q)_n}{(\frac{q^2}{a};q)_n}\left(\frac{b}{az}\right)^n$$

$$= \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty} \tag{2.7}$$

Using Rogers-Fine identity [1, (9.1.1), p. 223] in (2.7), we have

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n z^n}{(b;q)_n} = \sum_{n=0}^{\infty}\frac{(a;q)_n(azq/b;q)_n b^n z^n q^{n^2-n}(1-azq^{2n})}{(b;q)_n(z;q)_{n+1}}$$

$$+\frac{(1-\frac{q}{b})}{(1-\frac{q}{a})}\left(\frac{b}{az}\right)\sum_{n=0}^{\infty}\frac{(q^2/b;q)_n(q/z;q)_n\left(\frac{b}{az}\right)^n\left(\frac{q^2}{a}\right)^n q^{n^2-n}\left(1-\frac{q^{2n+2}}{az}\right)}{\left(\frac{q^2}{a};q\right)_n\left(\frac{b}{az};q\right)_{n+1}}$$

$$= \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty} \tag{2.8}$$

Taking $b = aq$ in (2.8) we have

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{1-aq^n} = \sum_{n=0}^{\infty}\frac{a^n z^n q^{n^2}(1-azq^{2n})}{(1-aq^n)(1-zq^n)} - \left(\frac{q}{az}\right)\sum_{n=0}^{\infty}\frac{q^{n^2+2n}\left(1-\frac{q^{2n+2}}{az}\right)}{\left(1-\frac{q^{n+1}}{a}\right)\left(1-\frac{q^{n+1}}{z}\right)(az)^n}$$

$$= \frac{(q;q)_\infty^2(az,q/az;q)_\infty}{(q,q/a,z,q/z;q)_\infty} \tag{2.9}$$

Replacing q by q^k and then replacing z by λq^i and a by μq^j in (2.9) we get (2.2).

3. Spaceial Cases

In this section we shall deduce certain interesting special cases of (2.1) and (2.2)

(i) Taking $\lambda = \mu$ and $i = j$ in (2.1) we get,

$$\sum_{n=0}^{\infty} \frac{\lambda^n q^{in}}{1-\lambda q^{kn+i}} = \sum_{n=0}^{\infty} \frac{q^{kn^2+2in}\lambda^{2n}(1+\lambda q^{kn+i})}{(1-\lambda q^{kn+i})}$$

$$= \frac{1}{(1-\lambda q^i)-} \frac{\lambda q^i(1-\lambda q^i)^2}{(1-\lambda q^{k+i})-} \frac{\lambda^2 q^{2i}(1-q^k)^2}{(1-\lambda q^{2k+i})-} \frac{\lambda q^{k+i}(1-\lambda q^{k+i})^2}{(1-\lambda q^{3k+i})-}$$
$$\frac{\lambda^2 q^{k+2i}(1-q^{2k})^2}{(1-\lambda q^{4k+i})-} \frac{\lambda q^{2k+i}(1-\lambda q^{2k+i})^2}{(1-\lambda q^{5k+i})-} \frac{\lambda^2 q^{2k+2i}(1-q^{3k})^2}{(1-\lambda q^{6k+i})-\ldots}. \tag{3.1}$$

(ii) Taking $\lambda = -1$, $\mu = 1$, $i = j = 2$, $k = 4$ in (2.1) and using [1; Entry (18.2.4), p. 397] we have

$$\Psi^2(q^4) = \sum_{n=0}^{\infty} \frac{(-)^n q^{2n}}{1-q^{4n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{4n^2+4n}(-)^n(1+q^{8n+4})}{(1-q^{8n+4})}$$

$$= \frac{1}{(1-q^2)+} \frac{q^2(1-q^2)^2}{(1-q^6)+} \frac{q^4(1-q^4)^2}{(1-q^{10})+} \frac{q^6(1-q^6)^2}{(1-q^{14})+} \frac{q^8(1-q^8)^2}{(1-q^{18})+\ldots} \tag{3.2}$$

(iii) Again, taking $\lambda = 1$, $\mu = -1$, $i = j = 1$, $k = 2$ in (2.1) and using [1; Entry (18.2.4), p. 397] we find,

$$\Psi^2(q^2) = \sum_{n=0}^{\infty} \frac{q^n}{1+q^{2n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(-)^n(1+q^{4n+2})}{(1-q^{4n+2})}$$

$$= \frac{1}{(1+q)-} \frac{q(1+q)^2}{(1+q^3)+} \frac{q^2(1-q^2)^2}{(1+q^5)-} \frac{q^3(1+q^3)^2}{(1+q^7)+}$$
$$\frac{q^4(1-q^4)^2}{(1+q^9)-} \frac{q^5(1+q^5)^2}{(1+q^{11})+} \frac{q^6(1-q^6)^2}{(1+q^{13})-\ldots} \tag{3.3}$$

Comparing (3.2) and (3.3) we have

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{1+q^{4n+2}} = \sum_{n=0}^{\infty} \frac{(-)^n q^{2n}}{(1-q^{4n+2})} \tag{3.3(a)}$$

(iv) Choosing $i = 1,\ \lambda = 1,\ k = 5$ in (3.1), we get

$$\sum_{n=0}^{\infty} \frac{q^n}{1-q^{5n+1}} = \sum_{n=0}^{\infty} \frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}$$

$$= \frac{1}{(1-q)-} \frac{q(1-q)^2}{(1-q^6)-} \frac{q^2(1-q^5)^2}{(1-q^{11})-} \frac{q^6(1-q^6)^2}{(1-q^{16})-} \frac{q^7(1-q^{10})^2}{(1-q^{21})-} \frac{q^{11}(1-q^{11})^2}{(1-q^{26})-} \frac{q^{12}(1-q^{15})^2}{(1-q^{31})-\ldots} \tag{3.4}$$

(v) Again, choosing $i = 2, \lambda = 1$ and $k = 5$ in (3.1), we find

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{5n+2}} = \sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})}$$

$$= \frac{1}{(1-q^2)-} \frac{q^2(1-q^2)^2}{(1-q^7)-} \frac{q^4(1-q^5)^2}{(1-q^{12})-} \frac{q^7(1-q^7)^2}{(1-q^{17})-} \frac{q^9(1-q^{10})^2}{(1-q^{22})-} \frac{q^{12}(1-q^{12})^2}{(1-q^{27})-\ldots} \tag{3.5}$$

(vi) Taking $\mu = \lambda, i = j$ in (2.2), we have

$$\sum_{n=-\infty}^{\infty} \frac{\lambda^n q^{in}}{1-\lambda q^{kn+i}} = \sum_{n=0}^{\infty} \frac{q^{kn^2+2ni}\lambda^{2n}(1+\lambda q^{kn+i})}{(1-\lambda q^{kn+i})}$$

$$-\sum_{n=0}^{\infty} \frac{q^{kn^2+n(2k-2i)+k-2i}\left(1+\frac{1}{\lambda}q^{kn+k-i}\right)\lambda^{-2n-4}}{\left(1-\frac{1}{\lambda}q^{kn+k-i}\right)} = \frac{(q^k;q^k)_\infty^2(\lambda^2q^{2i}, q^{k-2i}/\lambda^2;q^k)_\infty}{(\lambda q^i;q^k)_\infty^2(q^{k-i}/\lambda;q^k)_\infty^2} \tag{3.6}$$

(vii) Taking $k = 5i,\ \lambda = 1$ in (3.6) we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{in}}{1-q^{5in+i}} = \sum_{n=0}^{\infty} \frac{q^{5n^2i+2ni}(1+q^{5ni+i})}{(1-q^{5ni+i})} - \sum_{n=0}^{\infty} \frac{q^{5in^2+8in+3i}(1+q^{5ni+4i})}{(1-q^{5ni+4i})}$$

$$= \frac{(q^{5i};q^{5i})_\infty^2(q^{2i},q^{3i};q^{5i})_\infty}{(q^i,q^{4i};q^{5i})_\infty^2} \tag{3.7}$$

(viii) Again, choosing $k = 5i, \lambda = q^i$ in (3.6) we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2in}}{1-q^{5in+2i}} = \sum_{n=0}^{\infty} \frac{q^{5n^2i+4ni}(1+q^{5ni+2i})}{(1-q^{5ni+2i})} - \sum_{n=0}^{\infty} \frac{q^{5in^2+6in+i}(1+q^{5ni+3i})}{(1-q^{5ni+3i})}$$

$$= \frac{(q^{5i};q^{5i})^2_\infty (q^{4i},q^i;q^{5i})_\infty}{(q^{2i},q^{3i};q^{5i})^2_\infty} \tag{3.8}$$

(ix) Dividing (3.7) by (3.8) and comparing with [1; Corollary (6.2.6), p. 153] we get

$$\frac{\sum_{n=-\infty}^{\infty} \frac{q^{in}}{1-q^{5in+i}}}{\sum_{n=-\infty}^{\infty} \frac{q^{2in}}{1-q^{5in+2i}}} = \frac{\sum_{n=0}^{\infty} \frac{q^{5n^2i+2ni}(1+q^{5ni+i})}{(1-q^{5ni+i})} - \sum_{n=0}^{\infty} \frac{q^{5in^2+8in+3i}(1+q^{5ni+4i})}{(1-q^{5ni+4i})}}{\sum_{n=0}^{\infty} \frac{q^{5n^2i+4ni}(1+q^{5ni+2i})}{(1-q^{5ni+2i})} - \sum_{n=0}^{\infty} \frac{q^{5in^2+6in+i}(1+q^{5ni+3i})}{(1-q^{5ni+3i})}}$$

$$= \left\{ \frac{(q^{2i},q^{3i};q^{5i})_\infty}{(q^i,q^{4i};q^{5i})} \right\}^3$$

$$= \left\{ 1 + \frac{q^i}{1+} \frac{q^{2i}}{1+} \frac{q^{3i}}{1+ \ldots} \right\}^3 \tag{3.9}$$

For $i = 1$, (3.9) gives [1; Entry (4.4.1), p. 117].

(x) Taking $k = 4$, $i = 1$, $\lambda = 1$ in (3.6) and then using [1; (6.2.25), p. 151] we get

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}} = \sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{(1-q^{4n+1})} - \sum_{n=0}^{\infty} \frac{q^{4n^2+6n+2}(1+q^{4n+3})}{(1-q^{4n+3})}$$

$$= \frac{(q^4;q^4)^2_\infty (q^2;q^4)^2_\infty}{(q,q^3;q^4)^2_\infty} = \left\{ \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \right\}^2 = \psi^2(q)$$

$$= \left\{ \frac{1}{1-} \frac{q}{1+} \frac{q-q^2}{1-} \frac{q^3}{1+} \frac{q^2-q^4}{1-} \ldots \right\}^2 \tag{3.10}$$

(xi) Again, taking $k = 4$, $i = 2$, $j = 1$ in (2.2) and $\lambda = \mu = 1$ we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{4n+1}} = \sum_{n=0}^{\infty} \frac{q^{4n^2+3n}(1-q^{8n+3})}{(1-q^{4n+1})(1-q^{4n+2})} - \sum_{n=0}^{\infty} \frac{q^{4n^2+5n+1}(1-q^{8n+5})}{(1-q^{4n+2})(1-q^{4n+3})}$$

$$= \frac{(q^4;q^4)^2_\infty}{(q^2;q^4)^2_\infty} = \frac{(q^2;q^2)^2_\infty}{(q^2;q^4)^2_\infty} \tag{3.11}$$

Dividing (3.11) by (3.10) we get

$$\frac{\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{4n+1}}}{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{4n+1}}} = \frac{\sum_{n=0}^{\infty} \frac{q^{4n^2+3n}(1-q^{8n+3})}{(1-q^{4n+1})(1-q^{4n+2})} - \sum_{n=0}^{\infty} \frac{q^{4n^2+5n+1}(1-q^{8n+5})}{(1-q^{4n+2})(1-q^{4n+3})}}{\sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{(1-q^{4n+1})} - \sum_{n=0}^{\infty} \frac{q^{4n^2+6n+2}(1+q^{4n+3})}{(1-q^{4n+3})}}$$

$$= \frac{(q;q^2)_\infty^2}{(q^2;q^4)_\infty^4} = \left\{ \frac{(q;q^2)_\infty}{(q^2;q^4)_\infty^2} \right\}^2$$

making use of [1; (6.2.1), p. 150] we get

$$= \left\{ \frac{1}{1+} \frac{q}{1+} \frac{q+q^2}{1+} \frac{q^3}{1+} \frac{q^2+q^4}{1+} \dots \right\}^2 \tag{3.12}$$

(xii) Taking $\lambda = 1$, $i = 1$ and $k = 6$ in (3.6) we get

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{6n+1}} = \sum_{n=0}^{\infty} \frac{q^{6n^2+2n}(1+q^{6n+1})}{(1-q^{6n+1})} - \sum_{n=0}^{\infty} \frac{q^{6n^2+10n+4}(1+q^{6n+5})}{(1-q^{6n+5})}$$

$$= \frac{(q^6;q^6)_\infty (q^2;q^2)_\infty}{(q,q^5;q^6)_\infty^2} \tag{3.13}$$

Again, taking $\lambda = 1$, $i = 2$ and $k = 6$ in (3.6) we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{6n+2}} = \sum_{n=0}^{\infty} \frac{q^{6n^2+4n}(1+q^{6n+2})}{(1-q^{6n+2})} - \sum_{n=0}^{\infty} \frac{q^{6n^2+8n+2}(1+q^{6n+4})}{(1-q^{6n+4})}$$

$$= \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} \tag{3.14}$$

Dividing (3.13) by (3.14), we have

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{6n+1}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{6n+2}}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{6n^2+2n}(1+q^{6n+1})}{(1-q^{6n+1})} - \sum_{n=0}^{\infty} \frac{q^{6n^2+10n+4}(1+q^{6n+5})}{(1-q^{6n+5})}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{6n^2+4n}(1+q^{6n+2})}{(1-q^{6n+2})} - \sum_{n=0}^{\infty} \frac{q^{6n^2+8n+2}(1+q^{6n+4})}{(1-q^{6n+4})}}$$

$$= \frac{(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty^2 (q,q^5;q^6)_\infty^2} = \frac{(q^2;q^2)_\infty^2 (q^3;q^6)_\infty^2}{(q^3;q^3)_\infty^2 (q,q^5;q^6)_\infty^2} \tag{3.15}$$

(xiii) Taking $k = 6$, $i = 3$, $j = 1$, $\mu = \lambda = 1$ in (2.2) we find,

$$\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{6n+1}} = \sum_{n=0}^{\infty} \frac{q^{6n^2+4n}(1-q^{12n+4})}{(1-q^{6n+3})(1-q^{6n+1})} - \sum_{n=0}^{\infty} \frac{q^{6n^2+8n+2}(1-q^{12n+8})}{(1-q^{6n+3})(1-q^{6n+5})}$$

$$= \frac{(q^6;q^6)_\infty (q^2;q^2)_\infty}{(q,q^5;q^6)_\infty (q^3;q^6)_\infty^2} \tag{3.16}$$

Dividing (3.13) by (3.16) and using [1; Corollary (6.2.7), p. 154] we get

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{6n+1}}}{\displaystyle\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{1-q^{6n+1}}}=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{6n^2+2n}(1+q^{6n+1})}{(1-q^{6n+1})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+10n+4}(1+q^{6n+5})}{(1-q^{6n+5})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{6n^2+4n}(1+q^{12n+4})}{(1-q^{6n+3})(1-q^{6n+1})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+8n+2}(1-q^{12n+8})}{(1-q^{6n+3})(1-q^{6n+5})}}$$

$$=\frac{(q^3;q^6)_\infty^2}{(q,q^5;q^6)_\infty}=1+\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^4+q^8}{1+}... \tag{3.17}$$

(xiv) Taking $k=8$, $i=1$, $\lambda=1$ in (3.6)

$$\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{8n+1}}=\sum_{n=0}^{\infty}\frac{q^{8n^2+2n}(1+q^{8n+1})}{(1-q^{8n+1})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+14n+6}(1+q^{8n+7})}{(1-q^{8n+7})}$$

$$=\frac{(q^8;q^8)_\infty^2(q^2;q^4)_\infty}{(q,q^7;q^8)_\infty^2} \tag{3.18}$$

Again, taking $k=8$, $i=3$, $\lambda=1$ in (3.6) we get

$$\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{1-q^{8n+3}}=\sum_{n=0}^{\infty}\frac{q^{8n^2+6n}(1+q^{8n+3})}{(1-q^{8n+3})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+10n+2}(1+q^{8n+5})}{(1-q^{8n+5})}$$

$$=\frac{(q^8;q^8)_\infty^2(q^2;q^4)_\infty}{(q^3,q^5;q^8)_\infty^2} \tag{3.19}$$

Dividing (3.19) by (3.18) and then using [1; (6.2.38), p. 154] we get

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{1-q^{8n+3}}}{\displaystyle\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{8n+1}}}=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{8n^2+6n}(1+q^{8n+3})}{(1-q^{8n+3})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+10n+2}(1+q^{8n+5})}{(1-q^{8n+5})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{8n^2+2n}(1+q^{8n+1})}{(1-q^{8n+1})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+14n+6}(1+q^{8n+7})}{(1-q^{8n+7})}}$$

$$=\left\{\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty}\right\}^2$$

$$=\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+}\frac{q^5+q^{10}}{1+}...\right\}^2 \tag{3.20}$$

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