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A NOTE ON FRACTIONAL DERIVATIVE AND ITS APPLICATIONS

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Dedicated to Prof. K. Srinivasa Rao on his 75^{th} Birth Anniversary

Abstract: In this paper, starting from the historical developments of fractional calculus, certain results regarding fractional calculus have been discussed. These results have been further used to establish transformation formulae for ordinary hypergeometric series as well as for q-hypergeometric series.

Keywords and Phrases: Fractional derivative, fractional q-derivative, transformation formula, hypergeometric series, q-hypergeometric series.

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1. Introduction, Notations and Definitions

The generalized hypergeometric function ${}_pF_q(x)$ is defined as

$${}_pF_q\left[\begin{array}{l} a_1, a_2, ..., a_p; x \\ b_1, b_2, ..., b_q \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k ... (a_p)_k x^k}{(b_1)_k (b_2)_k ... (b_q)_k k!}. \tag{1.1}$$

When $q = p$, this series converges for $|x| < \infty$, but when $p = q + 1$, convergence occurs for $|x| < 1$. In (1.1) the Pochhammer symbol $(a)_k$ is defined by $(a)_0 = 1$ and for $k \geq 1$ by $(a)_k = a(a+1)...(a+k-1)$. However, for all integers k we write simply $(a)_k = \dfrac{\Gamma(a+k)}{\Gamma(a)}$.

We shall also use the notation,

$(a_1, a_2, ..., a_p)_k = (a_1)_k(a_2)_k...(a_p)_k$.
We shall also make use of the basic (or q-) hypergeometric function ${}_r\Phi_s$ which is defined by

$$ {}_r\Phi_s \left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_k z^k}{(q, b_1, b_2, ..., b_s; q)_k} \left\{(-1)^k q^{k(k-1)/2}\right\}^{1+s-r}, \quad (1.2) $$

where
$(a_1, a_2, ..., a_r; q)_k = (a_1; q)_k(a_2; q)_k...(a_r; q)_k$.

$$ (a; q)_k = \begin{cases} 1, & k = 0 \\ (1 - a)(1 - aq)...(1 - aq^{k-1}), & k \geq 1. \end{cases} $$

Also,

$$ (a; q)_\infty = \prod_{r=0}^{\infty}(1 - aq^r). $$

For a suitable function $y = f(x)$, Leibniz invented the notation $\frac{d^n y}{dx^n}$ for the n^{th} derivative of y provided that n is a positive integer. In 1695 L'Hospital asked Leibniz: "what if n be $\frac{1}{2}$? Leibniz replied," This is an apparent Paradox, from which, one day, useful consequences will be drawn." This original question of L'Hospital led to the name fractional calculus. Later the question become : can n be any number, fractional, irrational or complex ? Because the latter question was answered affirmatively, the name fractional calculus has become a misnomer and might better be called integration and differentiation to an arbitrary order.

(a) In 1819, the first mention of a derivative of arbitrary order appears in a text S.F. Lacroix devoted less than two pages of his 700 page text to this topic. Starting with $y = x^m$, m a positive integer, Lacroix easily developed

$$ \frac{d^n x^m}{dx^n} = \frac{m!}{m - n!} x^{m-n}, \quad m \geq n. \quad (1.3) $$

Using Legendre's symbol, for the generalized factorial, the Gamma function, he developed,

$$ \frac{d^\nu x^m}{dx^\nu} = \frac{\Gamma(m + 1)}{\Gamma(1 + m - \nu)} x^{m-\nu}, \quad (1.4) $$

where ν is an arbitrary number, fractional, irrational or complex.

(i) Taking $\nu = \frac{1}{2}$ and $m = 1$, it is easy to see from (1.4) that,

$$\frac{d^{1/2}}{dx^{1/2}}x = \frac{\Gamma(2)x^{1/2}}{\Gamma(2-1/2)} = \frac{x^{1/2}}{\Gamma(3/2)} = 2\sqrt{\frac{x}{\pi}}. \tag{1.5}$$

(ii) Taking $\nu = \frac{1}{2}$ and $m = 0$, we find from (1.4) that,

$$\frac{d^{1/2}}{dx^{1/2}}1 = \frac{\Gamma(1)x^{-1/2}}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi x}}. \tag{1.6}$$

(b) Liouville in 1832 gave the second definition of fractional calculus. His starting point was a definite integral,

$$I = \int_0^\infty u^{a-1}e^{-xu}du, \quad a > 0 \ \text{ and } \ x > 0. \tag{1.7}$$

Putting $xu = t$, it is easy to see that

$$I = x^{-a}\int_0^\infty t^{a-1}e^{-t}dt = x^{-a}\Gamma(a)$$

or

$$x^{-a} = \frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}e^{-xu}du. \tag{1.8}$$

Liouville operated with D^ν on both sides of (1.7) to obtain

$$\begin{aligned} D^\nu x^{-a} &= \frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}(-u)^\nu e^{-xu}du \\ &= \frac{(-1)^\nu}{\Gamma(a)}\int_0^\infty u^{a+\nu-1}e^{-xu}du \end{aligned} \tag{1.9}$$

Comparing (1.9) with (1.7) one finds,

$$D^\nu x^{-a} = \frac{(-1)^\nu\Gamma(a+\nu)}{\Gamma(a)}x^{-a-\nu}, \tag{1.10}$$

where $a > 0$.

(c) Starting with the n^{th} derivative of Cauchy's integral formula

$$D^n f(z) = \frac{n!}{2\pi i}\int_c \frac{f(w)}{(w-z)^{n+1}}dw \tag{1.11}$$

It was proved that

$$ {}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t) dt, \quad \Re\, \nu > 0. \tag{1.12} $$

The most used version of (1.12) is,

$$ {}_0D_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt, \quad \Re\, \nu > 0. \tag{1.13} $$

which is called Riemann-Liouville definition of fractional derivative in the form of integral.

(d) Leibniz's formula for the fractional derivative of the product of two functions is given by,

$$ D^{\nu}\{f(t)g(t)\} = \sum_{k=0}^{\infty} \binom{\nu}{k} \{D^k g(t)\}\{D^{\nu-k} f(t)\} $$

$$ \sum_{k=0}^{\infty} \frac{\Gamma(1+\nu)}{k!\Gamma(1+\nu-k)} \{D^k g(t)\}\{D^{\nu-k} f(t)\}, \tag{1.14} $$

where $\Re(\nu) > 0$ and $D^k\{g(t)\}$ is the ordinary derivative.

(e) Agarwal [1] in 1976 defined the fractional q-derivative,

$$ D_q^{\alpha} f(x) = (1-q)^{-\alpha} x^{-\alpha} \sum_{n=0}^{\infty} \frac{(q^{-\alpha};q)_n}{(q;q)_n} q^n f(xq^n) \tag{1.15} $$

and

$$ D_q^{\alpha} x^{\mu-1} = (1-q)^{-\alpha} \frac{(q^{\mu-\alpha};q)_\infty}{(q^{\mu};q)_\infty} x^{\mu-\alpha-1}. \tag{1.16} $$

The fractional q-derivative of the product of two functions is given as,

$$ D_q^{\lambda}(UV) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{(q^{-\lambda};q)_n}{(q;q)_n} D_q^n(V) D_q^{\lambda-n}\{U(xq^n)\}, \tag{1.17} $$

where $|x| < R$ and $U(x)$, $V(x)$ are two regular functions such that

$$ U(x) = \sum_{r=0}^{\infty} a_r x^r, \quad V(x) = \sum_{r=0}^{\infty} b_r x^r, $$

convergent in the regions $|x| < R_1$ and $|x| < R_2$ respectively and $R = \min(R_1, R_2)$.

2. Transformation formulae for ordinary hypergeometric series

In this section we shall make use of (1.4) and (1.14) in order to establish a transformation formulae for ordinary hypergeometric series.

(i) Let us consider a function $x^{\mu-1}e^x$. Taking the derivative of arbitrary order ν of this function we get

$$D^\nu(x^{\mu-1}e^x) = D^\nu \sum_{n=0}^{\infty} x^{\mu-1}\frac{x^n}{n!} = D^\nu \sum_{n=0}^{\infty} \frac{x^{\mu+n-1}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} D^\nu(x^{\mu+\nu-1}).$$

(Applying (1.4))

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\Gamma(\mu+n)}{\Gamma(\mu-\nu+n)} x^{\mu+n-\nu-1} = \frac{\Gamma(\mu)x^{\mu-\nu-1}}{\Gamma(\mu-\nu)} \sum_{n=0}^{\infty} \frac{(\mu)_n}{(\mu-\nu)_n}\frac{x^n}{n!}$$

$$= \frac{\Gamma(\mu)x^{\mu-\nu-1}}{\gamma(\mu-\nu)}\ {}_1F_1\left[\begin{matrix}\mu; x\\ \mu-\nu\end{matrix}\right]. \tag{2.1}$$

Again, applying (1.14), we obtain

$$D^\nu(x^{\mu-1}e^x) = \sum_{k=0}^{\infty} \frac{\Gamma(1+\nu)}{k!\Gamma(1+\nu-k)} D^k(e^x)D^{\nu-k}(x^{\mu-1})$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\Gamma(1+\nu)e^x}{\Gamma(1+\nu-k)}\frac{\Gamma(\mu)}{\Gamma(\mu-\nu+k)} x^{\mu-\nu+k-1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(-\nu)_k}{k!}\frac{\Gamma(\mu)e^x x^{\mu-\nu+k-1}}{\Gamma(\mu-\nu)(\mu-\nu)_k}$$

$$= \frac{\Gamma(\mu)e^x}{\Gamma(\mu-\nu)} x^{\mu-\nu-1}\ {}_1F_1\left[\begin{matrix}-\nu; -x\\ \mu-\nu\end{matrix}\right], \tag{2.2}$$

Comparing (2.1) and (2.2) we find

$${}_1F_1\left[\begin{matrix}\mu; x\\ \mu-\nu\end{matrix}\right] = e^x\ {}_1F_1\left[\begin{matrix}-\nu; -x\\ \mu-\nu\end{matrix}\right], \tag{2.3}$$

which is Kummer's first transformation.

We can also apply (1.14) by changing the functions. Thus we find,

$$D^\nu(x^{\mu-1}e^x) = \sum_{k=0}^{\infty} \frac{\Gamma(1+\nu)}{k!\Gamma(1+\nu-k)} D^k(x^{\mu-1})D^{\nu-k}(e^x)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(1+\nu)\Gamma(\mu)e^x x^{\mu-k-1}}{k!\Gamma(1+\nu-k)\Gamma(\mu-k)}$$

$$= e^x x^{\mu-1} \ {}_2F_0 \left[\begin{matrix} -\nu, 1-\mu; \frac{1}{x} \\ - \end{matrix} \right], \tag{2.4}$$

where $\mu > 1$ for the convergence of ${}_2F_0$ function.
Comparing (2.1) with (2.4) we get,

$$\Gamma(\mu) \ {}_1F_1 \left[\begin{matrix} \mu; x \\ \mu-\nu \end{matrix} \right] = e^x x^{\nu} \Gamma(\mu-\nu) \ {}_2F_0 \left[\begin{matrix} -\nu, 1-\mu; \frac{1}{x} \\ - \end{matrix} \right], \tag{2.5}$$

where $\mu > 1$.
Again, comparing (2.2) and (2.4) we have

$$\Gamma(\mu) \ {}_1F_1 \left[\begin{matrix} -\nu; -x \\ \mu-\nu \end{matrix} \right] = x^{\nu} \Gamma(\mu-\nu) \ {}_2F_0 \left[\begin{matrix} -\nu, 1-\mu; \frac{1}{x} \\ - \end{matrix} \right], \tag{2.6}$$

where $\mu > 1$.
Proceeding as above by taking different pair of functions, one can establish similar interesting transformations.

3. Transformation formulae for q-hypergeometric series

In this section we shall make use of (1.16) and (1.17) in order to establish a transformation formulae for q-series.
Let us consider the derivative,

$$D_q^{\lambda-\mu}\{x^{\lambda-1}e_q(x)\} = \sum_{n=0}^{\infty} D_q^{\lambda-\mu} \left\{ \frac{x^{n+\lambda-1}}{(q;q)_n} \right\}$$

Applying (1.16)

$$= \sum_{n=0}^{\infty} \frac{(1-q)^{\mu-\lambda}(q^{\mu+n};q)_\infty}{(q;q)_n(q^{\lambda+n};q)_\infty} x^{n+\mu-1}$$

$$= \frac{(1-q)^{\mu-\lambda}x^{\mu-1}(q^{\mu};q)_\infty}{(q^{\lambda};q)_\infty} \ {}_2\Phi_1 \left[\begin{matrix} q^{\lambda}; q; x \\ q^{\mu} \end{matrix} \right]. \tag{3.1}$$

Again, considering the same derivative and applying (1.17) we have

$$D_q^{\lambda-\mu}\{x^{\lambda-1}e_q(x)\} = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} \frac{(q^{\mu-\lambda};q)_n}{(q;q)_n} D_q^n\{e_q(x)\} D_q^{\lambda-\mu-n}\{(xq^n)^{\lambda-1}\} \tag{3.2}$$

Since,

$$D_q^n\{e_q(x)\} = \sum_{r=0}^{\infty} \frac{1}{(q;q)_r}(D_q^n x^r) = \sum_{r=0}^{\infty}(1-q)^{-n}\frac{(q^{1+r-n};q)_\infty x^{r-n}}{(q;q)_r(q^{1+r};q)_\infty}$$

$$= (1-q)^{-n}x^{-n}\sum_{r=n}^{\infty}\frac{x^r}{(q;q)_{r-n}}.$$

Putting $r+n$ for r we have

$$D_q^n\{e_q(x)\} = (1-q)^{-n}e_q(x). \tag{3.3}$$

Also,

$$D_q^{\lambda-\mu-n}\{x^{\lambda-1}q^{n\lambda-n}\} = q^{n\lambda-n}(1-q)^{\mu-\lambda+n}\frac{(q^{\mu+n};q)_\infty x^{\mu+n-1}}{(q^\lambda;q)_\infty}. \tag{3.4}$$

Putting these values in (3.2) we obtain

$$D_q^{\lambda-\mu}\{x^{\lambda-1}e_q(x)\} = (1-q)^{\mu-\lambda}x^{\mu-1}\frac{(q^\mu;q)_\infty}{(q^\lambda;q)_\infty}e_q(x)\ {}_1\Phi_1\left[\begin{matrix} q^{\mu-\lambda};q;xq^\lambda \\ q^\mu \end{matrix}\right] \tag{3.5}$$

Equating (3.1) and (3.5) we finally get,

$${}_2\Phi_1\left[\begin{matrix} 0,q^\lambda;q;x \\ q^\mu \end{matrix}\right] = e_q(x)\ {}_1\Phi_1\left[\begin{matrix} q^{\mu-\lambda};q;xq^\lambda \\ q^\mu \end{matrix}\right] \tag{3.6}$$

which is q-analogue of Kummer's first transformation of ${}_1F_1$ series.

Proceeding in the same way by taking proper choice of functions one can establish a number of useful transformations.
For details one can consult [2], [3] and [4] also.

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