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HIERARCHIES OF PALINDROMIC SEQUENCES IN THE SYMMETRIC GROUP S_n

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Dedicated to Prof. K. Srinivasa Rao on his 75^{th} ***Birth Anniversary***

Abstract: A new property of the Symmetric group, S_n, arises when each element is assigned a unique place value, which enables the ordering of the elements (numerically). It is shown that the differences between successive elements of this ordered, place-value assigned symmetric group S_n, give rise to a palindromic sequence $(\mathcal{P}_k)^{\beta}_{1\leq k\leq n!-1}$. We define the family of palindromic sequences, associated to S_n. Sieving out a given number at a time in the maximal palindromic sequence of the group, S_n, of length $n! - 1$, results in a hierarchy of palindromic sequences, ending with a single element, which will be the central element of $(\mathcal{P}_k)^{\beta}_{1\leq k\leq n!-1}$. Consequences of the concept of place value ordering of the elements of S_n, are presented in this article.

1. Introduction

Symmetric groups have been extensively studied in the field of abstract Algebra. A symmetric group is the set of all the permutations of the indices $\{1, 2, \ldots, n\}$, denoted by S_n. As the number of permutations of indices $\{1, 2, \ldots, n\}$ are $n!$, S_n is a finite group of order $n!$.

Definition 1. Let $<$ be a lexicographic ordering on the elements of S_n. Consider the two permutations of S_n, $\sigma = (a_1\ a_2\ \cdots\ a_n)$ and $\pi = (b_1\ b_2\ \cdots\ b_n)$. We say $\sigma < \pi$, if there exists an element $i \in [n]$, such that $\forall\ j < i$, $\sigma_{(j)} = \pi_{(i)}$ and

$\sigma_{(i)} < \pi_{(j)}$, where $\sigma_{(i)}$ and $\pi_{(j)}$ denotes the i-th and j-th element in permutation σ and π respectively, and $[n] \equiv \{1, 2, \ldots, n\}$.
Consider the smallest non-trivial symmetric group S_3, which is the set of six elements, written down in lexicographic order :

$$\sigma_1 = \begin{pmatrix} 1\ 2\ 3 \\ 1\ 2\ 3 \end{pmatrix} \equiv (1\ 2\ 3);\ \sigma_2 = \begin{pmatrix} 1\ 2\ 3 \\ 2\ 3\ 1 \end{pmatrix} \equiv (2\ 3\ 1);\ \sigma_3 = \begin{pmatrix} 1\ 2\ 3 \\ 3\ 1\ 2 \end{pmatrix} \equiv (3\ 1\ 2);$$

$$\sigma_4 = \begin{pmatrix} 1\ 2\ 3 \\ 2\ 1\ 3 \end{pmatrix} \equiv (2\ 1\ 3);\ \sigma_5 = \begin{pmatrix} 1\ 2\ 3 \\ 3\ 2\ 1 \end{pmatrix} \equiv (3\ 2\ 1);\ \sigma_6 = \begin{pmatrix} 1\ 2\ 3 \\ 1\ 3\ 2 \end{pmatrix} \equiv (1\ 3\ 2);$$

Assigning the lexicographic ordering on the permutations of $\{1, 2, 3\}$ leads to the 'standard' ordering [1]:

$$1\ 2\ 3, \quad 1\ 3\ 2, \quad 2\ 1\ 3, \quad 2\ 3\ 1, \quad 3\ 1\ 2, \quad 3\ 2\ 1.$$

The following is a definition for assigning place-value to the permutations belonging to S_n:

Definition 2. $f_B : S_n \longrightarrow \mathbb{N}$,

$$f_B(\sigma) = \sum_{i=1}^{n} \sigma_{(i)}\ B^{n-i}, \qquad \text{for some} \quad B \in \mathbb{N}, \quad B \geq 2.$$

f_B will be referred to as the *place-valued* function. This definition enables arithmetic operations to be made on the permutations. Differences between successive/adjacent place-value ordered permutations lead to a sequence of numbers. For instance, in the case of S_3, with B = 10, the 5 differences between the six specifically ordered permutations give rise to the sequence:

$$9,\ 81,\ 18,\ 81,\ 9$$

which is a *palindromic* sequence.
The paper is outlined as follows. In section 2., the definition of a palindromic sequence $(\mathcal{P}_k)^{\beta}_{1\leq k\leq n!-1}$ is presented along with the main theorem and its proof. Section 3., contains the alternate proof of the main theorem, which is free of induction. In section 4., the properties of the palindromic sequences which constitute a family are presented. Section 5. contains the sieving methods and the ways to obtain the hierarchies of palindromic sequences, from the original palindromic sequence of length $n! - 1$.

2. Main Theorem

The following are essential definitions required to prove the main Theorem.

Definition 3. Let $\mathcal{S}_{(n!)} = \{\sigma_1, \sigma_2, \ldots, \sigma_{n!}\}$ will be denoted the place-value assigned, lexicographically ordered symmetric group.

Definition 4. The palindromic sequence $(\mathcal{P}_k)^{\beta}_{1\leq k\leq n!-1}$, on a particular value of $B = \beta$ is associated with $\mathcal{S}_{(n!)}$, given by the elements:

$$\mathcal{P}_k \;=\; \sum_{1\leq i\leq n} (b_i - a_i) \times \beta^{n-i},$$

where $\quad a_i \in \sigma_j \quad$ and $\quad b_i \in \sigma_{j+1}, \quad$ for all $\quad 1 \leq j \leq (n! - 1)$.
Following is the statement of the main theorem :

Theorem 5. The differences between the successive elements of $\mathcal{S}_{(n!)}$ generate a palindromic sequence $(\mathcal{P}_k)^{\beta}_{1\leq k\leq n!-1}$ of length $n! - 1$.

Proof. The Proof has to be preceded by two more definitions, a Claim and a Lemma which are stated below:

Definition 6. Given a permutation $\sigma = (a_1,\ a_2,\ \cdots,\ a_n)$, consider the function

$$f\ :\ \mathcal{S}_{(n!)} \to \mathcal{S}_{(n!)},$$

where $$f(\sigma) \;=\; f(a_1\, a_2\, \cdots\, a_n) \;=\; h(a_1)\, h(a_2) \cdots\, h(a_n).$$

Definition 7. Consider the '*elemental*' function

$$h : \{1\ 2\ \cdots n\} \to \{1\ 2\ \cdots n\},$$

where $\quad h(1) \to n,\ h(2) \to n-1, \cdots, h(i) \to n-i+1,\ \cdots, h(n-1) \to 2, h(n) \to 1.$
The following claim provides an insight to the mapping done by the function f over the elements of the set $\mathcal{S}_{(n!)}$.

Claim 8. $f(\sigma_j) = \sigma_{n!-j+1}$.

Proof. To prove the calim, let $P(j)$ be the statement that $f(\sigma_j) = \sigma_{n!-j+1}$. The proof by induction on the index of permutations in $\mathcal{S}_{(n!)}$ follows:

Basic Step: $f(\sigma_1) = f(1\ 2\ 3\ \ldots\ n-1\ n) = h(1)h(2)\ldots h(n-1)h(n)$
$= n\ (n-1)\ \ldots\ 3\ 2\ 1 = \sigma_{n!}$, by definition.

Inductive Step: Let $P(j)$ be true for all j with $2 \leq j \leq k-1$. Consider

$$\sigma_{k-1} = (s_1,\ s_2,\ \ldots,\ s_{m-1},\ s_m,\ s_{m+1},\ \ldots,\ s_n).$$

The next permutation in lexicographic order, *a la* [2], is

$$\sigma_k = (s'_1,\ s'_2,\ \ldots,\ s'_{m-1},\ s'_m,\ s'_{m+1} \ldots,\ s'_n).$$

Since σ_{k-1} and σ_k are adjacent permutations, $s_i = s'_i$, for all $1 \le i \le (m-1)$, it follows that $h(s_i) = h(s'_i)$. Now s'_m is the smallest element amongst $(s_{m+1},\ s_{m+2},\ \ldots,\ s_n)$, such that $s'_m > s_m$, so we get $h(s'_m) < h(s_m)$, and since

$$s'_{m+1} < s'_{m+2} < \ldots s'_n \quad \Rightarrow \quad h(s'_{m+1}) > h(s'_{m+2}) > \ldots > h(s'_n),$$

and it follows that $h(\sigma_k)$ gives just the permutation, prior to $h(\sigma_{k-1})$. Now by induction hypothesis,

$$h(\sigma_{k-1}) = \sigma_{n!-(k-1)+1} \quad \Rightarrow \quad h(\sigma_k) = \sigma_{n!-k+1}.$$

Hence the proof of the claim.

Consider any two permutations

$$\sigma_p = (a_1,\ a_2,\ \ldots,\ a_n), \qquad \sigma_q = (b_1,\ b_2,\ \ldots,\ b_n)$$

for all $p < q \le n!$, from the left end of the set $\mathcal{S}_{(n!)}$. Let

$$f(\sigma_p) = \sigma_{n!-p+1} = (d_1,\ d_2,\ \ldots,\ d_n), \qquad f(\sigma_q) = \sigma_{n!-q+1} = (c_1,\ c_2,\ \ldots,\ c_n),$$

be the p-th and the q-th elements of the same set, but from the right end.

Lemma 9. *$(b_i - a_i) = (d_i - c_i)$, $\forall\ 1 \le i \le n$.*

Proof. By the Claim made above,

$$f(\sigma_p) = f(a_1,\ a_2,\ \ldots,\ a_n) = h(a_1)h(a_2)\ldots h(a_n) = (d_1\ d_2 \ldots\ d_n) = \sigma_{n!-p+1}.$$

Similarly,

$$f(\sigma_q) = f(b_1,\ b_2,\ \ldots,\ b_n) = h(b_1)h(b_2)\ldots h(b_n) = (c_1\ c_2 \ldots\ c_n) = \sigma_{n!-q+1}.$$

Note that $n! - p + 1 > n! - q + 1$, $\forall\ p < q \le n!$. Let $a_i = \ell$ and $b_i = r$ where $\ell, r \in [n]$. Now we get

$$d_i - c_i = h(a_i) - h(b_i) = (n - \ell + 1) - (n - r + 1) = r - \ell = b_i - a_i.$$

Thus, the proof of the lemma.
Now a proof of the main theorem follows:

The difference between the decimal place values of the permutations σ_j and σ_{j+1}, from the left end of the set $\mathcal{S}(n!)$, are given as:

$$\sum_{1\leq i\leq n} (b_i - a_i) \times \beta^{n-i}.$$

Similarly, the difference between the decimal place values of the permutations $\sigma_{n!-j+1}$ and $\sigma_{n!-(j+1)+1}$, from the right end of $\mathcal{S}(n!)$, are given as:

$$\sum_{1\leq i\leq n} (d_i - c_i) \times \beta^{n-i} \quad \equiv \quad \sum_{1\leq i\leq n} (b_i - a_i) \times \beta^{n-i},$$

since by the Lemma,

$$(b_i - a_i) = (d_i - c_i) \quad \forall \quad 1 \leq i \leq n.$$

So, the same number occurs, when we take the difference between two successive permutations, from either end of $\mathcal{S}_{(n!)}$, and consequently, the resultant sequence of length $(n! - 1)$ is *palindromic.*

3. An Alternate Proof

The main theorem can also be proved in an alternate setting of the 'elemental' function, defined as the *index* function below:

Definition 10. An index function

$$h' : \mathcal{S}_{(n)} \to \mathcal{S}_{(n)}, \text{ such that } h'(\sigma) = \pi,$$

where $\forall\ i \in \{1, 2, \ldots, n\}, \quad \pi_{(i)} = n + 1 - \sigma_{(i)}$.

Definition 11. Predecessor of any element σ of the set $\mathcal{S}_{(n!)}$, is another element π, and it can be written as $p(\sigma) = \pi$, where $\sigma = \sigma_j$, $\pi = \sigma_{j-1}$.
Since $\sigma < \pi \Leftrightarrow h'(\sigma) > h'(\pi)$,

$$h'(\sigma_1)h'(\sigma_2)\ldots h'(\sigma_{n!}) = \sigma_{n!}\sigma_{n!-1}\ldots\sigma_1$$

and for every j, $\quad h'(\sigma_j) = \sigma_{n!+1-j}$. It follows that

$$f_\beta(\sigma) - f_\beta(p(\sigma)) = f_\beta(h'(p(\sigma))) - f_\beta(h'(\sigma)).$$

consequently, the sequence $(\mathcal{P}_k)^\beta_{1\leq k\leq n!-1}$ is *Palindromic.*

4. Properties of Palindromic Sequences: $(\mathcal{P}(k))^\beta_{1\leq k\leq n!-1}$

Every term in the palindromic sequence, $(\mathcal{P}(k))^\beta_{1\leq k\leq n!-1}$, is a multiple of $(\beta - 1)$, since

$$\begin{aligned}
\sum_{1\le i\le n}(\sigma(i)-p(\sigma(i)))\times\beta^{n-i} &= \sum_{1\le i\le n}(\sigma(i)-p(\sigma(i)))\times((\beta-1)+1)^{n-i},\\
&= \sum_{1\le i\le n}(\sigma(i)-p(\sigma(i)))\ \mathrm{mod}\ (\beta-1),\\
&= \sum_{1\le i\le n}(\sigma(i)-\sum_{1\le i\le n}p(\sigma(i))\ \mathrm{mod}\ (\beta-1),\\
&= 0\\
\text{as}\quad \sum_{1\le i\le n}\sigma(i) &= \sum_{1\le i\le n}p(\sigma(i))
\end{aligned}$$

Remark 12. The palindromic sequence, $(\mathcal{P}_k)^{\beta}_{1\le k\le n!-1}$, for $(\mathcal{S})_{(n!)}$, contains the palindromic sequence $(\mathcal{P}_k)^{\beta}_{1\le k\le(n-1)!-1}$ for $(\mathcal{S})_{((n-1)!)}$ at both ends, and in between there is a new palindromic sub-sequence of length $(n-2)\times(n-1)!+1$.

Remark 13.

$$f_\beta(\sigma)+f_\beta(h(\sigma))=(n+1)\sum_{i=1}^{n}\beta^{n-i}=(n+1)(\frac{\beta^n-1}{\beta-1})$$

which is an n digit number, with each digit as $(n+1)$.

Remark 14.

$$f_\beta(\sigma)+f_\beta(h'(\sigma))=\sum_{i=1}^{n}(\sigma_{(i)}+\sigma_{(n-i+1)})\beta^{n-i}$$

which is a palindromic number of n digits.

Definition 15. The family of palindromic sequences, contains the sequences obtained by taking differences between successive elements of the set $\mathcal{S}_{(n!)}$, denoted by $(\mathcal{P}_k)^{B\ge 2}_{1\le k\le n!-1}$, where each palindromic sequence $(\mathcal{P}_k)^{\beta}_{1\le k\le n!-1}$ is subject to the specific value of B $=\beta$, as defined above.

5. Sieving Procedure

Definition 16. An *irreducible* sequence, is a sequence whose elements have no common multiplicative factor.

It is interesting to note the regular repetitive nature of the elements in the palindromic sequence $(\mathcal{P}_k)^{\beta}_{1\le k\le n!-1}$, where by sieving / canceling the repeating elements in an ordered / sequential manner:

- A given number $(a\ b\ c\ d\ e) \in \mathcal{S}_5$, with $a \neq b \neq c \neq d \neq e$ is such that the sum of the first and the last numbers is a constant number, for example

  $$(1\ 2\ 3\ 4\ 5) + (5\ 4\ 3\ 2\ 1) = (6\ 6\ 6\ 6\ 6).$$

  For, every $(a\ b\ c\ d\ e)$ has $((6-a)\ (6-b)\ (6-c)\ (6-d)(6-e))$ as its corresponding *reversed / complementary* number such that each individual digit adds to 6. Arranging $(n!/2)$ in an ascending order and taking the difference in the place-value assigned permutations gives a sequence of numbers. The differences between adjacent complementary numbers gives their reverse ordered sequence of the same set of numbers. Thus, we have a palindromic sequence of numbers of length $n!/2$.

- The palindromic sequence for $\mathcal{S}_3$ can be extended to a palindromic sequence for $\mathcal{S}_4$ by the addition of the digit 4, to either end of each one of the six elements

  $$(1\ 2\ 3),\ (1\ 3\ 2),\ (2\ 1\ 3),\ (2\ 3\ 1),\ (3\ 1\ 2),\ (3\ 2\ 1),$$

  to get:

  $$(1\ 2\ 3\ 4),\ (1\ 3\ 2\ 4),\ (2\ 1\ 3\ 4),\ (2\ 3\ 1\ 4),\ (3\ 1\ 2\ 4),\ (3\ 2\ 1\ 4)$$

  and

  $$(4\ 1\ 2\ 3),\ (4\ 1\ 3\ 2),\ (4\ 2\ 1\ 3),\ (4\ 2\ 3\ 1),\ (4\ 3\ 1\ 2),\ (4\ 3\ 2\ 1).$$

  The differences between successive elements of these sets, give rise now to the palindromic sub-sequences

  $$90,\ 810,\ 180,\ 810,\ 90 \qquad \text{and} \qquad 9,\ 81,\ 18,\ 81,\ 8.$$

  It is to be noted that all the elements in these sequences are multiple of 90 and 9, respectively. Hence, one may call the sequence $1,\ 9,\ 2,\ 9,\ 1$ an *irreducible* palindromic sequence, whenever the common factor is factored out, from the sequence, and in the present case both the palindromic sequences give rise to the same *irrseq* or I.S. However, when 4 is inserted between 2 and 3 in the 10^1 position of the elements of $\mathcal{S}_{(3)}$ giving rise to the following six elements of $\mathcal{S}_{(4)}$ and their corresponding differences:

  $$(1\ 2\ 4\ 3),(1\ 3\ 4\ 2),(2\ 1\ 4\ 3),(2\ 3\ 4\ 1),(3\ 1\ 4\ 2),(3\ 2\ 4\ 1) \Leftrightarrow 99, 801, 198, 801, 99,$$

This now results in the *irrseq* 11, 89, 22, 89, 11. Finally, when 4 is inserted between 1 and 2, in the 10^2 position of $\mathcal{S}_{(3)}$, it gives rise to the remaining six elements of $\mathcal{S}_{(4)}$:

$$(1\,4\,2\,3),\ (1\,4\,3\,2),\ (2\,4\,1\,3),\ (2\,4\,3\,1),\ (3\,4\,1\,2),\ (3\,4\,2\,1) \Leftrightarrow 9, 981, 18, 981, 9,$$

resulting in the *irrseq* 1, 109, 2, 109, 1.

In the case of the group S_4, the 24 permutations may be written down as the 'ordered' sequence:

$$(1\ 2\ 3\ 4),\quad (1\ 2\ 4\ 3),\quad (1\ 3\ 2\ 4),\quad (1\ 3\ 4\ 2),\quad (1\ 4\ 2\ 3),\quad (1\ 4\ 3\ 2),$$

$$(2\ 1\ 3\ 4),\quad (2\ 1\ 4\ 3),\quad (2\ 3\ 1\ 4),\quad (2\ 3\ 4\ 1),\quad (2\ 4\ 1\ 3),\quad (2\ 4\ 3\ 1),$$

$$(3\ 1\ 2\ 4),\quad (3\ 1\ 4\ 2),\quad (3\ 2\ 1\ 4),\quad (3\ 2\ 4\ 1),\quad (3\ 4\ 1\ 2),\quad (3\ 4\ 2\ 1),$$

$$(4\ 1\ 2\ 3),\quad (4\ 1\ 3\ 2),\quad (4\ 2\ 1\ 3),\quad (4\ 2\ 3\ 1),\quad (4\ 3\ 1\ 2),\quad (4\ 3\ 2\ 1).$$

The differences between consecutive assigned place values of these permutations are:

$$9,\ 81,\ 18,\ 81,\ 9,\ 702,\ 9,\ 171,\ 27,\ 72,\ 18,\ \underline{693},$$

$$18,\ 72,\ 27,\ 171,\ 9,\ 702,\ 9,\ 81,\ 18,\ 81,\ 9\ \ \text{– Seq. I}$$

This is a palindromic sequence of 23 elements or of length 23, with 693 as the pivotal central element. It is interesting to note that the absolute differences between successive elements of this palindromic sequence of length 23, give rise to a new palindromic sequence of length 22. Needless to say, this procedure of taking absolute differences between successive elements of a palindromic sequence can be continued until the final difference element is 0.
From the present Seq.I – viz. the palindromic sequence of <u>length 23</u> – the differences between adjacent elements of the sequence give rise to a palindromic sequence of length 22. Continuing this procedure, in a step-by-step fashion, gives rise to 22 palindromic sequences, which are given below (the first sequence alone is in two lines):

$$72,\ 63,\ 63,\ 72,\ 693,\ 693,\ 162,\ 144,\ 45,\ 54,\ \underline{675},$$

$$\underline{675},\ 54,\ 45,\ 144,\ 162,\ 693,\ 693,\ 72,\ 63,\ 63,\ 72.$$

$$9,\ 0,\ 9,\ 621,\ 0,\ 531,\ 18,\ 99,\ 9,\ 621,\ \underline{0},\ 621,\ 9,\ 99,\ 18,\ 531,\ 0,\ 621,\ 9,\ 0,\ 9$$

$$9, 9, 612, 621, 531,\ 513,\ 81,\ 90,\ 612,\ \underline{621},\ \underline{621},\ 612,\ 90,\ 81,\ 513,\ 531, 621, 612, 9, 9$$

$$0,\ 603,\ 9,\ 90,\ 18,\ 432,\ 9,\ 522,\ 9,\ \underline{0},\ 9,\ 522,\ 9,\ 432,\ 18,\ 90,\ 9,\ 603,\ 0$$

603, 594, 81, 72, 414, 423, 513, 513, $\underline{9}$, $\underline{9}$, 513, 513, 423, 414, 72, 81, 594, 603

9, 513, 9, 342, 9, 90, 0, 504, $\underline{0}$, 504, 0, 90, 9, 342, 9, 513, 9

504, 504, 333, 333, 81, 90, 504, $\underline{504}$, $\underline{504}$, 504, 90, 81, 333, 333, 504, 504

0, 171, 0, 252, 9, 414, 0, $\underline{0}$, 0, 414, 9, 252, 0, 171, 0

171, 171, 252, 243, 405, 414, $\underline{0}$, $\underline{0}$, 414, 405, 243, 252, 171, 171

0, 81, 9, 162, 9, 414, $\underline{0}$, 414, 9, 162, 9, 81, 0

81, 72, 153, 153, 405, $\underline{414}$, $\underline{414}$, 405, 153, 153, 72, 81

9, 81, 0, 252, 9, $\underline{0}$, 9, 252, 0, 81, 9

72, 81, 252, 243, $\underline{9}$, $\underline{9}$, 243, 252, 81, 72

9, 171, 9, 234, $\underline{0}$, 234, 9, 171, 9

162, 162, 225, $\underline{234}$, $\underline{234}$, 225, 162, 162

0, 63, 9, $\underline{0}$, 9, 63, 0

63, 54, $\underline{9}$, $\underline{9}$, 54, 63

9, 45, $\underline{0}$, 45, 9

36, $\underline{45}$, $\underline{45}$, 36

9, $\underline{0}$, 9

9, 9

0

An interesting feature of the 23-element palindromic sequence is that all the elements are multiples of 9 and so the following sequence maybe called as the *Irreducible* palindromic sequence:

1, 9, 2, 9, 1, 78, 1, 19, 3, 8, 2, $\underline{77}$, 2, 8, 3, 19, 1, 78, 1, 9, 2, 9 1 – Seq. II

As in the case of Seq. I, it is straightforward to obtain another hierarchy of sequences, of length reduced by one unit at a time, when we take the absolute differences between the adjacent elements of Seq. II.

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