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ON GENERALIZED η-DUALS OF SOME SEQUENCE SPACES

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Abstract: P. Chandra and B.C. Tripathy [5] have generalized the notion of the Köthe-Toeplitz dual of sequence spaces on introducing the concept of η-dual of order r for $r \geq 1$ of sequence spaces.
Ansari and Gupta [3] have generalized the notion of the Köthe-Toeplitz dual of sequence spaces on introducing the concept of η-dual of order r for $0 < r \leq 1$.
We have defined and determined the η-dual of some sequence spaces for $r > 0$ and have established their perfectness in relation the η-dual for $r > 0$.

Keywords and Phrases: Dual space, perfect space, η dual, convergent sequence, l_r space, bounded variation, cesaro summable sequence.

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1. Introduction

The idea of dual sequence space was introduce by Köthe and Toeplitz [10], whose main result with α-duals. An account of the duals of sequence spaces is found in Köthe [11]. The different type of duals of sequence spaces are found in cook [2]. Maddox [4], Kamthan and Gupta and many others. In this paper w, c, c_0, l_s, l_p, l_∞, v, v_∞, b_v, w_p denoted the space of all, convergent, null absolutely summable, p-absolutely summable, bounded convergent series, series with bounded partial sum, bounded variation sequence, p-Cesaro summable sequence spaces respectively.

The α-dual of a subset E of w is defined as

$$E^\alpha = \{\langle a_n \rangle \in w : (a_n x_n) \in l_1 \forall (x_n) \in E\}$$

If we replace l_1 by v and v_∞ in the above definition then we shall get β and γ-duals of E respectively.

Basarir [7], Lascarides [9], Maddox [4] and other have studied results involving α-dual and β-dual of different sequence spaces and their properties.

let E be a non-empty subset of w and $r \geq 1$ then the η-dual of E is defined by P. Chandra and B.C. Tripathy [5] as,

$$E^\eta = \{\langle a_n \rangle \in w : (a_n x_n) \in l_r \forall (x_n) \in E\}$$

A non-empty subset E of w is said to be perfect or η-reflexive. If $E^{\eta\eta} = E$. Taking $r = 1$ in the above definition we get the α-dual of E.

Let E be a non-empty subset of w and $0 \leq r \leq 1$ then η-dual of E is defined by Ansari and Gupta [3] as

$$E^\eta = \{(a_n) \in w : (a_n x_n) \in l_r \forall (x_n) \in E\}$$

A non-empty subset E of w is said to be perfect or η-reflexive if $E^{\eta\eta} = E$. Taking $r = 1$ in the above definition we get the α-dual of E.

Let E be a non-empty subset of w and $r > 0$ then η-dual of E, we defined as

$$E^\eta = \{(a_n) \in w : (a_n x_n) \in l_r \forall (x_n) \in E\}$$

A non-empty subset E of w is said to be perfect or η-reflexive if $E^{\eta\eta} = E$. Taking $r = 1$ in the above definition we get the α-dual of E. .2cm

2. Main Results

In this section we fined the η-dual of some sequences spaces and establish whether they are perfect or not relative to η-dual.

Lemma

(i) E^η is a linear subspace of w for every $E \subset W$.

(ii) $E \subset F$ implies $E^\eta \supset F^\eta \forall E, F \subset W$.

(iii) $E^{\eta\eta} = (E^\eta)^\eta \supset E \subset W$.

(iv) $(U_j E_j)^\eta = \bigcup_j E_j^\eta$ for every family $\{E_j\}$ with $E_j \subset W$ for all $j \in N$, where N is the set of Natural Number.

Theorem 2.1: $l_r^\eta = l_\infty$, $l_\infty^\eta = l_r$ and the space l_r and l_∞ are perfect spaces, where $r > 0$.

Proof. First we shall show that $l_r^\eta = l_\infty$, where l_r^η defined as,

$$l_r^\eta = \left\{ (a_n) \in w : \sum_{n=1}^{\infty} |a_n x_n|^r < \infty \text{ for every } (x_n) \in l_r \right\}$$

Let $(a_n) \in l_\infty$ and $(x_n) \in l_r$

$$\Rightarrow \sup_{n\geq 1} |a_n| < \infty \text{ and } \sum_{n=1}^{\infty} |x_n|^r < \infty$$

therefore

$$\sum_{n=1}^{\infty} |a_n x_n|^r = \sum_{n=1}^{\infty} |a_n|^r |x_n|^r \leq \left(\sup_{n\geq 1} |a_n|^r \right) \left(\sum_{n=1}^{\infty} |x_n|^r \right) < \infty$$

$\Rightarrow \sum_{n=1}^{\infty} |a_n x_n|^r$ converges $\forall (x_n) \in l_r$. Hence we have $a_n \in l_r^\eta$ i.e. $l_\infty \subseteq l_r^\eta$ for the converse.

Let $(a_n) \notin l_\infty$ there exist (a_n) will have a subsequence (a_{n_i}) such that $a_{n_i} > i^s$ for some fixed real number $s > \frac{1}{r}$ where is a positive integer

Define a sequence (a_n)

$$x_n = \begin{cases} \frac{1}{i^s} & \text{if } n = n_i \\ 0 & \text{if } n \neq n_i \end{cases}$$

then

$$\left(\sum_{n=1}^{\infty} |x_n|^r \right) = \left(\sum_{i=1}^{\infty} (i^{-s})^r \right) = \sum_{n=1}^{\infty} \frac{1}{i^{sr}} < \infty$$

Since $sr > 1 \Rightarrow (x_n) \in l_r$

$$\therefore \quad \sum_{n=1}^{\infty} |a_n x_n|^r \geq \sum_{n=1}^{\infty} |\frac{1}{i^s}.i^s|^r = \infty$$

$$\Rightarrow (a_n x_n) \in l_r$$

then $(a_n) \notin l_r^\eta$
$\Rightarrow l_r^\eta \subset l_\infty$
Hence $l_r^\eta = l_\infty$ similarly we can prove that $l_r^\eta = l_r$

Since $l_\infty^{\eta\eta} = (l_\infty^\eta)^\eta = l_r^\eta = l_\infty$ and $l_r^{\eta\eta} = (l_r^\eta)^\eta = l_\infty^\eta = l_r$
Hence the spaces l_∞ and l_r are perfect.

Definition 1: Let σ denote the space of all eventually alternating sequence i.e. if $(x_n) \in \sigma$ then there exist $n_0 \in N$ such that $x_n = -x_{n+1} \forall n > n_0$ it is well known that

$$b\nu_0 = b\nu \cap C_0$$

Theorem 2.2: $\sigma^\eta = l_r$ and σ is not perfect.
Proof. Since $\sigma \subset l_\infty$, then by lemma (ii) $l_\infty^\eta \subset \sigma^\eta$, we have

$$l_r \subset \sigma^\eta \quad \text{by therorem 2.1} \tag{1}$$

let $(a_n) \in \sigma^\eta$ then $\sum_{n=1}^{\infty} |a_n x_n|^r < \infty$, for every $(x_n) \in \sigma$.
Let us define a sequence $(x_n) \in \sigma$ such that, $x_{2n-1} = 1 = -x_{2n} \forall n \in N$. Then, $\sum_{n=1}^{\infty} |x_n a_n|^r = \sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow (a_n) \in l_r$. Therefore,

$$\sigma^\eta \subseteq l_r \tag{2}$$

From (1) and (2), we have

$$\sigma^\eta = l_r$$

Since $\sigma^{\eta\eta} = (\sigma^\eta)^\eta = l_r^\eta = l_\infty \neq \sigma$. Hence σ is not perfect.

Theorem 2.3: $C_0^\eta = C^\eta = l_r$ and the sequence spaces C_0 and C are not perfect.
Proof. Since $C_0 \subset l_\infty$, by lemma (ii) $l_\infty^\eta \subset C_0^\eta$ and by theorem 2.1. $l_r = l_\infty^\eta$

$$\Rightarrow l_r \subseteq C_0^\eta \tag{3}$$

Again, let $(a_n) \in C_0^\eta$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n x_n|^r < \infty \quad \text{for every} \quad (x_n) \in C_0$$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n^r z_n| < \infty, \quad \text{for every} \quad (z_n) = (x_n^r) \in C_0$$

$$\Rightarrow (a_n^r) \in C_0^r = l_1$$

$$\Rightarrow (a_n) \in l_r$$

Thus

$$C_0^{\eta} \subseteq l_r \tag{4}$$

From (3) and (4)

$$C_0^{\eta} = l_r \tag{5}$$

Since, $C_0 \subset C \subset l_\infty$ by lemma (ii), $l_\infty^{\eta} \subset C^{\eta} \subset C_0^{\eta}$ from theorem 2.1 and result (5)

$$l_r = l_\infty^{\eta} = C^{\eta} = C_0^{\eta} = l_r$$

$$\Rightarrow C_0^{\eta} = C^{\eta} = l_r$$

$$C_0^{\eta\eta} = (C_0^{\eta})^{\eta} = l_r^{\eta} = l_\infty \neq C_0$$

and

$$C^{\eta\eta} = (C^{\eta})^{\eta} = l_r^{\eta} = l_\infty \neq C$$

Hence, the sequence spaces C_0 and C are not perfect with respect to η-dual for $r > 0$.

Theorem 2.4: $(C_0 \cap l_\infty)^{\eta} = (C \cap l_\infty)^{\eta} = l_r$ and the spaces $C_0 \cap l_\infty$ and $C \cap l_\infty$ are not perfect.

Proof. Since $C_0 \subset l_\infty$, therefore $C_0 \cap l_\infty = C_0$

$$\Rightarrow (C_0 \cap l_\infty)^{\eta} = C_0^{\eta} = l_r \quad \text{by Theorem 2.3} \tag{6}$$

Also, $C \subset l_\infty$, therefore $C \cap l_\infty = C$

$$\Rightarrow (C \cap l_\infty)^{\eta} = C^{\eta} = l_r \;\; \text{by theorem 2.3} \tag{7}$$

From (6) and (7)

$$(C_0 \cap l_\infty)^{\eta} = (C \cap l_\infty)^{\eta} = l_r$$

therefore

$$(C_0 \cap l_\infty)^{\eta\eta} = ((C_0 \cap l_\infty)^{\eta})^{\eta} = l_r^{\eta} = l_\infty \neq C_0 \cap l_\infty$$

and

$$(C \cap l_\infty)^{\eta\eta} = ((C \cap l_\infty)^{\eta})^{\eta} = l_r^{\eta} = l_\infty \neq C \cap l_\infty$$

Hence, the spaces $C_0 \cap l_\infty$ and $C \cap l_\infty$ are not perfect.

Theorem 2.5: $(b\nu)^{\eta} = l_r = (b\nu_0)^{\eta}$ and the spaces b_ν and $b\nu_0$ are not perfect.

Proof. Since,

$$b\nu = \left\{ x \in w : \lim_{m\to\infty} \sum_{i=1}^{m} |x_{i+1} - x_i| \;\; \text{exist} \right\}$$

and

$$b\nu_0 = \left\{x \in b\nu : \lim_{m\to\infty} x_m = 0\right\} = b\nu \cap C_0$$

therefore $b\nu \subset l_\infty$, by lemma (ii) $l_\infty^\eta \subset (b\nu)^\eta$

$$\Rightarrow l_r \subset (b\nu)^\eta \quad \text{by theorem 2.1} \tag{8}$$

Again, let $(a_n) \in (b\nu)^\eta$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n x_n|^r < \infty \forall (x_n) \in (b\nu)$$

Consider the sequence (x_n) defined by, $x_n = 1, \quad \forall n \in N$.
Then $(x_n) \in b\nu$ and $\sum_{n=1}^{\infty} |a_n x_n|^r = \sum_{n=1}^{\infty} |a_n|^r$ converges. $(a_n) \in l_r$ hence,

$$(b\nu)^\eta \subset l_r \tag{9}$$

From (8) and (9)

$$(b\nu)^\eta = l_r$$

Now we shall show that $(b\nu_0)^\eta = l_r$, therefore $b\nu_0 \subset b\nu \subset l_\infty$, by lemma (ii)

$$l_\infty^\eta \subset (b\nu)^\eta \subset (b\nu_0)^\eta$$

$$\Rightarrow l_r \subset (b\nu_0)^\eta \quad \text{from theorem 2.1}$$

Again, let $(a_n) \in (b\nu_0)^\eta$ but $(a_n) \notin l_r$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n x_n|^r < \infty \forall (x_n) \in b\nu_0 \quad \text{and} \quad \sum_{n=1}^{\infty} |a_n|^r = \infty$$

$\Rightarrow$ We can fined a sequence (k_n) of natural number (with $k_1 = 1$) such that

$$\sum_{i=k_n}^{k_{n+1}-1} |a_i|^r < n^r \quad \text{for all} \quad n = 1, 2, 3, ...$$

Now define the sequence (x_n) as $x_i = n^{-1}$ if $k_n \leq i \leq k_{n+1} - 1$ for all n = 1,2, 3,...
Then,

$$\sum_{n=1}^{\infty} |\triangle x_n| = \sum_{n=1}^{\infty} \sum_{i=k_n}^{k_{n+1}-1} |\triangle x_i|, \quad \text{where} \quad \triangle x_n = x_{n+1} - x_n$$

$$= \sum_{n=1}^{\infty} |x_{k_{n+1}} - x_{n_k}|$$

$$= \sum_{n=1}^{\infty} \left| \frac{1}{n+1} - \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty$$

Hence, $(x_n) \in b\nu_0$ further

$$\sum_{n=1}^{\infty} |a_n x_n|^r = \sum_{n=1}^{\infty} \sum_{i=k_n}^{k_{n+1}-1} |a_i x_i|^r = \sum_{n=1}^{\infty} \sum_{i=k_n}^{k_{n+1}-1} |a_i|^r . |x_i|^r$$

$$\sum_{n=1}^{\infty} \sum_{i=k_n}^{k_{n+1}-1} |a_i|^r > \sum_{n=1}^{\infty} \frac{1}{n^r} n^r = \infty$$

which is a contradiction.
Thus, $(b\nu_0) \subseteq l_r$, hence, $(b\nu_0)^\eta = l_r$.
Thus, we get

$$(b\nu)^\eta = l_r = (b\nu_0)^\eta$$

therefore

$$(b\nu)^{\eta\eta} = ((b\nu)^\eta)^\eta = l_\infty \neq b\nu$$

and

$$(b\nu_0)^{\eta\eta} = ((b\nu_0)^\eta)^\eta = l_r^\eta \neq b\nu_0$$

Hence, the sequence spaces $b\nu$ and $b\nu_0$ are not perfect.

Definition: Let w_p be denoted p-cesaro summable sequence space and defined as,

$$w_p = \left\{ (x_n) \in w : \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{m} |x_n - l|^p = 0 \text{ for some l and} 0 < p < \infty \right\}$$

Theorem 2.6: $(w_p \cap l_\infty)^\eta = l_r$ and the space $w_p \cap l_\infty$ is not perfect.
Proof. Since $w_p \cap l_\infty \subset l_\infty$, by lemma (ii), $l_\infty^\eta \subset (w_p \cap l_\infty)^\eta$

$$l_r \subset (w_p \cap l_\infty)^\eta \quad \text{(therefore } \; l_\infty = l_r \; \text{ from theroem 2.1)} \tag{10}$$

let $(a_n) \notin l_r$, then $\sum_{n=1}^{\infty} |a_n|^r = \infty$
Consider the sequence (x_n) defined as, $(x_n) = 1$ for all $n \in N$,
Then $(x_n) \in w_p \cap l_\infty$ but

$$\sum_{n=1}^{\infty} |a_n x_n|^r = \sum_{n=1}^{\infty} |a_n|^r |x_n|^r$$

$$\sum_{n=1}^{\infty} |a_n x_n|^r = \infty$$

Hence $(a_n) \notin (w_p \cap l_\infty)^\eta$, therefore

$$(w_p \cap l_\infty)^\eta \subseteq l_r \tag{11}$$

From (10) and (11)

$$(w_p \cap l_\infty)^\eta = l_r$$

therefore

$$(w_p \cap l_\infty)^{\eta\eta} = ((w_p \cap l_\infty)^\eta)^\eta = l_r^\eta = l_\infty \neq w_p \cap l_\infty$$

Hence, the space $w_p \cap l_\infty$ is not perfect.

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