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PARTITION THEORETIC INTERPRETATIONS OF CERTAIN IDENTITIES OF ROGERS-RAMANUJAN TYPE

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Abstract: In this paper, certain identities of Rogers-Ramanujan type, taken from Slater's paper, have been interpreted by making use of additive number theory.

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1. Introduction, notations and definitions

As usual, for α and q complex numbers with $|q| < 1$, let

$$(\alpha; q)_0 = 1, \quad (\alpha; q)_n = \prod_{r=0}^{n-1}(1 - \alpha q^r), \quad \text{for} \quad n \in N$$

and

$$(\alpha; q)_\infty = \prod_{r=0}^{\infty}(1 - \alpha q^r).$$

Sometimes, for brevity, we write

$$(a_1; q)_n(a_2; q)_n ...(a_r; q)_n = (a_1, a_2, ..., a_r; q)_n$$

There are following three type of identities available in the literature
(i) Series = Product

(ii) Series = Series
(iii) Product = Product

The most famous identities of first kind are Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{1.2}$$

Identities of the type, Series = Product, are called Rogers-Ramanujan type identities.

These two identities have a curious history. They were first proved by L.J. Rogers in 1894 [3] in a paper which was completely ignored. The identities were rediscovered by Ramanujan some time before 1913. In 1917, Ramanujan searched out the paper of Rogers containing the above identities from the library of Trinity College, London.

In 1917, these two identities were rediscovered and proved independently by German Mathematician Issai Schuv. There are now many different proofs of the identities available in the literature.

There are many identities of second type available in the Notebooks of Ramanujan. One best example of such type of identities is Rogers-Fine identity,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \sum_{n=0}^{\infty} \frac{(a;q)_n (azq/b;q)_n b^n z^n q^{n^2-n}(1-azq^{2n})}{(q;q)_n (z;q)_{n+1}} \tag{1.3}$$

[Andrwes, G.E. and Berndt, B.C. 2; (9.1.1), p. 223]

There are some useful identities of third type. One example of such identity is Euler's identity, namely

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} \tag{1.4}$$

P.A. MacMahon interpreted Rogers-Ramanujan identities (1.1) & (1.2) in the following ways respectively,

(i) If $D_2(n)$ denotes the number of partitions of n in which any two summands differ by at least 2 and $P(S_2, n)$ denotes the number of partitions of n into parts congruent to 1 or 4 modulo 5, then

$$D_2(n) = P(S_2, n) \tag{1.5}$$

[Andrews, G.E. 1; (14-1-1), p. 175]

(ii) If $D_2'(n)$ is the number of partitions of n in which any two summands differ by at least 2 and all summands are greater than 1 and $P(T_2, n)$ is the number of partitions of n into parts congruent to 2 or 3 modulo 5, then

$$D_2'(n) = P(T_2, n) \tag{1.6}$$

[Andrews, G.E. 1; (14-1-3), p. 176]

The main aim of this paper is give additive number theoretic interpretations of certain identities of type (1.1) and (1.2) selected from L.J. Slater's paper [4].

We shall make use of following identities due to Slater [4] which is the modified form can be expressed as

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} = \frac{1}{(q^2;q^4)_\infty}. \tag{1.7}$$

[Slater, L.J., 4; (7), p. 153]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^{2n+2};q^2)_\infty}{(q^2;q^2)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{1.8}$$

[Slater, L.J., 4; (16), p. 153]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^{2n+3};q^2)_\infty}{(q^2;q^2)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{1.9}$$

[Slater, L.J., 4; (17), p. 153]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^{2n+2};q^2)_\infty}{(q^2;q^2)_n} = \frac{1}{(q,q^4;q^5)_\infty}. \tag{1.10}$$

[Slater, L.J., 4; (20), p. 154]

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(-q^{2n+2};q)_\infty}{(q^2;q^2)_n} = \frac{1}{(q^2,q^3,q^4,q^5;q^7)_\infty}. \tag{1.11}$$

[Slater, L.J., 4; (31), p. 155]

$$\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^{2n+1};q)_\infty}{(q^2;q^2)_n}=\frac{1}{(q,q^3,q^4,q^6;q^7)_\infty}. \tag{1.12}$$

[Slater, L.J., 4; (32), p. 155]

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}(-q^{2n+1};q)_\infty}{(q^2;q^2)_n}=\frac{1}{(q,q^2,q^5,q^6;q^7)_\infty}. \tag{1.13}$$

[Slater, L.J., 4; (33), p. 155]

A Useful Result

Let us consider a partition of a positive integer m into n parts such that no part is less than a and least difference between any two parts is d. We can represent such type of partition as
$m = a_1 + a_2 + ... + a_n$
where
$a_1 \geq a$
$a_2 \geq a + d$
$a_3 \geq a + 2d$
.
.
.
$a_n \geq a + (n-1)d$
Thus, we have $a_1 + a_2 + ... + a_n \geq na + \frac{n(n-1)d}{2}$.
We have, now a partition

$$m-\left\{na+\frac{n(n-1)d}{2}\right\}=(a_1-a)(a_2-a-d)...(a_n-a-(n-1)d) \tag{1.14}$$

Thus, we find the following result;
For a partition of a positive integer m into n parts such that no part is less than a and minimal difference between two parts is d, there exists another partitions of $m-\left\{na+\frac{n(n-1)d}{2}\right\}$ as

$$m-\left\{na+\frac{n(n-1)d}{2}\right\}=(a_1-a)(a_2-a-d)...(a_n-a-(n-1)d)$$

We shall make use of (1.14) in our onward analysis.

2. Main Theorems

In this section, we shall establish following theorems

Theorem 1. The number of partitions of a positive integer m into parts $\equiv$2 (mod 4) equals the number of partitions of m into distinct even parts.

Theorem 2. The number of partitions of a positive integer m into parts $\equiv$2 or 3 (mod 5) equals the number of partitions of m into distinct odd parts each $\geq$3 and distinct even parts each $\geq$2n+2, where n is the number of odd parts in the partition.

Theorem 3. The number of partitions of a positive integer m into parts $\equiv$2 or 3 (mod 5) equals the number of partitions of m into distinct even parts and distinct odd parts each $\geq$2n+3, where n is the number of even parts in the partition.

Theorem 4. The number of partitions of a positive integer m into parts $\equiv$1 or 4 (mod 5) equals the number of partitions of m into distinct odd parts and distinct even parts each $\geq$2n+2, where n is the number of odd parts in the partition.

Theorem 5. The number of partitions of a positive integer m into parts $\equiv \pm 2, \pm 3$ (mod 7) equals the number of partitions of m into distinct even parts each $\geq$4 with minimal difference 4 and also certain distinct parts each $\geq$2n+2, where n is the number of even parts each $\geq$4 with minimal difference 4 in the partition.

Theorem 6. The number of partitions of a positive integer m into parts $\equiv \pm 1, \pm 3 (\text{mod } 7)$ equals the number of partitions of m into distinct even parts each $\geq$4 with minimal difference 4 and also certain distinct parts each $\geq$2n+1, where n is the number of even parts each $\geq$4 with minimal difference also 4 in the partition.

Theorem 7. The number of partitions of a positive integer m into parts $\equiv \pm 1, \pm 2 (\text{mod } 7)$ equals the number of partitions of m into even parts with minimal difference 4 and certain distinct parts each $\geq$2n+1, where n is the number of even parts with minimal difference also 4 in the partition.

3. Proofs of theorem 1 to 7

(i) Proof of theorem 1.

If we take a =2, d = 2 in (1.14) we get

$$na + \frac{n(n-1)d}{2} = n(n+1)$$

Now, let us consider a partition of m into distinct even parts i.e.

$$m = a_1 + a_2 + ... + a_n \tag{3.1}$$

where
$a_1 \geq 2, a_2 \geq 4, a_3 \geq 6, ..., a_n \geq 2n$
So,
$a_1 + a_2 + ... + a_n \geq n(n+1)$
Thus, we have a partition of

$$m - n(n+1) = (a_1 - 2) + (a_2 - 4) + ... + (a_n - 2n) \tag{3.2}$$

in where there at most n even parts. It is generated by $\dfrac{q^{n(n+1)}}{(q^2;q^2)_n}$.
The series $\sum_{n=0}^{\infty} \dfrac{q^{n(n+1)}}{(q^2;q^2)_n}$ gives the total number of partitions of the type (3.2).
Since there is one-one correspondence between (3.1) and (3.2), so the above series also gives the total number of partitions of type (3.1).
Hence, by an appeal of the identity (1.7), theorem 1 is proved.

(ii) Proof of theorem 2.

Taking a = 3, d = 2 in (1.14) we have

$$na + \frac{n(n-1)}{2}d = 3n + n(n-1) = n(n+2)$$

.
Now, let us consider a partition of the positive integer m into distinct odd parts each ≥3 and contain distinct even parts each ≥2n+2, where n is the number of odd parts in the partition i.e.

$$m = a_1 + a_2 + ... + a_n + d_1 + d_2 + ..., \text{(ascending order)} \tag{3.3}$$

where $d_1 \geq 2n + 2$
$a_1 \geq 3$
$a_2 \geq 3 + 2$
$a_3 \geq 3 + 4$
.
.
.
$a_n \geq 3 + (n-1)d,$
So, $a_1 + a_2 + ... + a_n \geq 3n + n(n-1) = n(n+2).$
Now, there exists a new partition,

$$m - n(n+2) = (a_1 - 3) + (a_2 - 5) + ... + (a_n - 2n - 1) + d_1 + d_2 + ... \tag{3.4}$$

which has at most n even parts.

The partition of type (3.4) is generated by $\dfrac{q^{n(n+2)}(-q^{2n+2};q^2)_\infty}{(q^2;q^2)_n}$.

Thus $\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q^{2n+2};q^2)_\infty}{(q^2;q^2)_n}$ gives total number of partitions of type (3.4). Since there is 1-1 correspondence between (3.3) and (3.4), so this series also gives total number of partitions of type (3.3). Hence by an appeal of identity (1.8) theorem 2 is proved.

(iii) Proof of theorem 3.

Let us consider a partition of a positive integer m into distinct even parts and contain distinct odd parts ≥2n+3, where n is the number of even parts in the partition i.e.

$$m = a_1 + a_2 + ... + a_n + o_1 + o_2 + o_3 + ..., \tag{3.5}$$

where

$$\left.\begin{array}{l} a_1 \geq 2, \\ a_2 \geq 4, \\ a_3 \geq 6, \\ . \\ . \\ . \\ a_n \geq 2n \end{array}\right\} o_1, o_2, o_3, ..., \text{are distinct odd parts and } o_1 \geq 2n+3.$$

Now, we have

$a_1 + a_2 + ... + a_n \geq n(n+1)$.

There exists a new partition

$$m - n(n+1) = (a_1 - 2) + (a_2 - 4) + ... + (a_n - 2n) + o_1 + o_2 + ..., \tag{3.6}$$

which has at most n even parts. (3.6) is generated by $\dfrac{q^{n(n+1)}(-q^{2n+3};q^2)_\infty}{(q^2;q^2)_n}$ and so $\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^{2n+3};q^2)_\infty}{(q^2;q^2)_n}$ gives total number of partitions of type (3.6). Due to 1-1 correspondence between (3.5) and (3.6), this series also gives the total number of partitions of type (3.5). So, by an appeal of the identity (1.9), theorem (3) is proved.

(iv) Proof of theorem 4.

If we take a = 1, and d = 2 in (1.14), we get

$$na + \frac{n(n-1)}{2}d = n^2$$

Let us consider a partition of the positive integer m into distinct odd parts and contain distinct even parts each ≥2n+2, where n is the number of distinct odd parts in the partition i.e.

$$m = a_1 + a_2 + ... + a_n + e_1 + e_2 + e_3 + ..., \text{(ascending order)} \tag{3.7}$$

where $a_1 \geq 1, a_2 \geq 3, ..., a_n \geq 2n-1$ and $e_1, e_2, e_3, ...,$ are distinct even parts such that $e_1 \geq 2n+2$.

$$a_1 + a_2 + ... + a_n \geq 1 + 3 + 5 + ... + (2n-1) = n^2$$

Now, we have a new partition

$$m - n^2 = (a_1 - 1) + (a_2 - 3) + ... + \{a_n - (2n-1)\} + e_1 + e_2 + ..., \tag{3.8}$$

which has at most n even parts. It is generated by $\dfrac{q^{n^2}(-q^{2n+2};q^2)_\infty}{(q^2;q^2)_n}$ and the series $\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^{2n+2};q^2)_\infty}{(q^2;q^2)_n}$ gives the total number of partitions of type (3.8). It also gives total number of partitions of type (3.7), due to 1-1 correspondence between (3.7) & (3.8). Hence by an appeal of the identity (1.10), theorem (4) is proved.

(v) Proof of theorem 5.

If we take a = 4 and d = 4 in (1.14), we find

$$na + \frac{n(n-1)}{2}d = 4n + 2n^2 - 2n = 2n(n+1).$$

Let us consider a partition of a positive integer m into distinct even parts each ≥ 4 with minimal difference 4 and contain distinct parts each ≥2n+2, where n is the number of even parts of the partition each ≥4 with minimal difference 4, i.e.

$$m = a_1 + a_2 + ... + a_n + d_1 + d_2 + d_3 + ... + \text{(ascending order)} \tag{3.9}$$

where $a_1 \geq 4, a_2 \geq 8, ..., a_n \geq 4n$ and
$d_1, d_2, d_3, ...,$ are all distinct and $d_1 \geq 2n+2$.

We have
$a_1 + a_2 + ... + a_n \geq 2n(n+1)$. Thus, we find a new partition,

$$m - 2n(n+1) = (a_1 - 4) + (a_2 - 8) + ... + (a_n - 4n) + d_1 + d_2 + ... \tag{3.10}$$

which has at most n even parts. It is generated by $\dfrac{q^{2n(n+1)}(-q^{2n+2}; q)_\infty}{(q^2; q^2)_n}$.
The series $\displaystyle\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(-q^{2n+2}; q)_\infty}{(q^2; q^2)_n}$ gives total number of partitions of type (3.10) and also total number of partition of type (3.9), due to one to one correspondence between (3.9) and (3.10).
So, by an appeal of identity (1.11), theorem (5) is proved.

(vi) Proof of theorem 6.

Let us consider a partition of the positive integer m into parts each ≥4 with minimal difference also 4 and contain distinct parts ≥2n+1, where n is number of parts each ≥4 with minimal difference 4 is the partition i.e.

$$m = a_1 + a_2 + ... + a_n + d_1 + d_2 + d_3 + ...(\text{ascending order}) \tag{3.11}$$

where $a_1 \geq 4, a_2 \geq 8, ..., a_n \geq 4n$ and $d_1, d_2, d_3, ...,$ are distinct parts such that $d_1 \geq 2n + 1$.
So $a_1 + a_2 + ... + a_n \geq 2n(n+1)$.
We have, now find a new partition,

$$m - 2n(n+1) = (a_1 - 4) + (a_2 - 8) + ... + (a_n - 4n) + d_1 + d_2 + ... \tag{3.12}$$

which has at most n even parts. It is generated by $\dfrac{q^{2n(n+1)}(-q^{2n+1}; q)_\infty}{(q^2; q^2)_n}$.
The series $\displaystyle\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(-q^{2n+1}; q)_\infty}{(q^2; q^2)_n}$ gives total number of partitions of type (3.12) and also total number of partition of type (3.11) due to one to one correspondence between (3.11) and (3.12).
Hence, by an appeal of the identity (1.12), theorem (6) is proved.

Proof of theorem 7.

If we take a = 2 and d = 4 in (1.14), we find

$$na + \frac{n(n-1)}{2}d = 2n + 2n^2 - 2n = 2n^2$$

Let us consider a partition of a positive integer m into distinct even parts each with minimal difference 4 and contain distinct parts each $\geq$2n+1, where n is the number of even parts of the partition each $\geq$4 with minimal difference 4, i.e.

$$m = a_1 + a_2 + ... + a_n + d_1 + d_2 + d_3 + ...(\text{ascending order}) \tag{3.13}$$

where $a_1 \geq 2, a_2 \geq 6, ..., a_n \geq 4n - 2$ and $d_1, d_2, d_3, ...$ are all distinct such that $d_1 \geq 2n + 1$.
Now, we have,
$a_1 + a_2 + ... + a_n \geq 2n^2$.
Thus, there exists a new partition

$$m - 2n^2 = (a_1 - 2) + (a_2 - 6) + ... + (a_n - 4n + 2) + d_1 + d_2 + ... \tag{3.14}$$

which has at most n even parts. It is generated by $\dfrac{q^{2n^2}(-q^{2n+1}; q)_\infty}{(q^2; q^2)_n}$

The series $\displaystyle\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^{2n+1}; q)_\infty}{(q^2; q^2)_n}$ gives total number of partitions of type (3.14) and also total number of partition of type (3.13), due to one to one correspondence between (3.13) and (3.14).
Hence, theorem (7) is proved by an appeal of identity (1.13).

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