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EXPLORING INFINITY MATHEMATICALLY

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Abstract: Infinity has been explored by developing a newer approach to arrive at mathematically logical and paradox-free conclusions.

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1. Introduction

Infinite/Infinity has been a topic of discussion since ancient times. There has been discussion on physical as well as abstract infinities by considering them in largest and smallest possible forms but it has often ended in quagmires / paradoxes due to one or the other reason [1, 2, 3].

In the present investigation, abstract infinity has been explored. Such an infinity has its genesis in the abstraction of natural numbers by way of dissociating them from the process of counting of finite number of things present in the given collection, and then, associating them with the operation of successive addition of one, which can be continued for ever. Thus $N = \{1, 2, 3, ...\}$ came to be recognized as the first infinite set having infinity as its cardinal number. However, it may be added that this infinity is not a number itself but a characteristic of the infinite set N.

In fact, the cardinal number of every infinite set will be infinity. Thus, with a view to explore such infinities, it will be necessary to first define the cardinal number of the set N. This, in turn, will also facilitate in arriving at mathematically logical and paradox-free conclusions.

2. Cardinality of the set N

For working out a mathematical definition of the cardinal number of the set N, note that the concept of infinity was introduced to keep logically thinking minds at rest, with the understanding that it could be reached, nay, approached by visualizing step-by-step movement from 1 to 2, 2 to 3, and so on, to continue for ever. Thus, consider a finite set of n natural numbers $\{1, 2, 3, ...n\}$ and note that its cardinal number will be 'n', or say $|n|$, to differentiate it from the natural number 'n'. Hence, the cardinal number of the set N, say $\bar{N}$, can be mathematically defined as given by $\bar{N} = \lim_{n\to\infty} |n|$, and N being the first set, can be taken to represent an infinity of order one.

This procedure will now be applied to the infinite set of squares (Galileo's demonstration) and the sets **Z, Q** and **R** to discover some of their mathematically logical and interesting features.

3. Cardinality of the infinite set of squares (Galileo's Demonstration)

Consider a finite set of n natural numbers $\{1, 2, 3, ...n\}$ and note that the corresponding finite set of squares viz. $\{1^2, 2^2, 3^2, ..., [\sqrt{n}]^2\}$ will have its cardinal number given by $[\sqrt{n}]$, denoting integral part of $\sqrt{n}$. Accordingly, the cardinal number of the corresponding infinite set of squares will be $\lim_{n\to\infty} [\sqrt{n}]$ which, like $\bar{N} = \lim_{n\to\infty} |n|$, will also be infinite but it can be taken to represent infinity of order 1/2 or say $\bar{N}^{1/2}$. Besides this, one can also visualize, as follows.

(i) The infinite set of squares is less denser than N.

(ii) If one has to approach infinity by stepping onto the squares, then it will be faster than stepping onto the natural numbers.

Here, it is pertinent to note that the above observations are about abstract numbers but the words approach, less denser, and faster used therein belong to the physical world.

4. Cardinality of the set Z of integers

For discussing the cardinality of **Z**, note that it contains **N** also. Hence, **Z** will be considered sans **N**; the single element zero can also be excluded from **Z** without any loss of generality. Thus, while considering such a duly modified **Z** corresponding to $\{1, 2, 3, ...n\}$, consider the set $\{-1, -2, -3, ... -n\}$ and note that its cardinal number will also be $|n|$. Hence the cardinal number of the duly modified **Z** will also be $\lim_{n\to\infty} |n| = \bar{N}$ i.e. infinity of order one. Hence, while considering the cardinalities of **Q** and **R**, zero and the negative numbers will be excluded from them.

5. Cardinality of the set Q of rational numbers

The set **Q** contains **Z** and hence while considering its cardinality, only fractions of the form a/b will be considered, where a and b are positive integers such that $a \neq$

b and $b \neq 1$. Accordingly, consider a finite set of n natural numbers $\{1, 2, 3, ...n\}$ and form the following (n - 1) possible subsets of the fractions of the form a/b from them, as elaborated below.

Firstly, form the subset $\{1/2, 1/3, 1/4, ...1/n\}$ by taking $a = 1$ and $b = 2$ to n and note that its cardinal number C_1 will be (n - 1) i.e. $C_1 = n - 1$.

Secondly, form the subset $\{2/3, 2/5, 2/7, ...2/n, 3/2, 5/2, 7/2, ...n/2\}$ by taking $a = 2$ and $b = 3$ to n, subject to $gcd(2, b) = 1$ and then extending it by including their reciprocals. The cardinal number C_2 of such a subset will be less than $2(n-2)$ due to the condition $gcd(2, b) = 1$ i.e. $C_2 < 2(n - 2)$.

Thirdly, form the subset $\{3/4, 3/5, 3/7, ...3/n, 4/3, 5/3, 7/3, ...n/3\}$ by taking $a = 3$ and $b = 4$ to n, subject to $gcd(3, b) = 1$, and then extending it by including their reciprocals. The cardinal number C_3 of such a subset will be less than $2(n-3)$ due to the condition $gcd(3, b) = 1$ i.e. $C_3 < 2(n - 3)$.

Proceeding this way, the cardinal number C_{n-1} of the last $(n - 1)^{th}$ subset $\{(n - 1)/n, n/(n - 1)\}$ will be 2 i.e. $C_{n-1} = 2$.

Thus, the required cardinal number C of the finite set of fractions formed from 1, 2, 3, ... n will be given by $C = C_1 + C_2 + C_3 + ... + C_{n-1}$ which will be less than $(n - 1) + 2[(n - 2) + (n - 3) + ... + 1] = (n - 1) + 2.(n - 2)(n - 1)/2 = (n - 1)^2$ i.e. $C < (n - 1)^2$.

Accordingly, the cardinal number of the duly modified set **Q** will be $< \lim_{n\to\infty} [(n-1)^2] =< \lim_{n\to\infty} n^2 = \bar{N}^2$ i.e. infinity of order two.

From the above investigation, one can also visualize, as follows.

(i) The duly modified set **Q** is more denser than **N**.

(ii) If one has to approach infinity by stepping onto the fractions then it will be slower than stepping onto the natural numbers.

6. Cardinality of the set R of real numbers

The set **R** contains **Q** and hence while considering its cardinality, only irrational numbers of the form $(m)^{1/k}$ where $m, k > 1$, will be considered; the exclusion of transcendental numbers from **R** has been explained in the next section 7.

Accordingly, consider a finite set of n natural numbers $\{1, 2, 3, ...n\}$ and form the following $(n - 1)$ possible subsets of irrational numbers of the said form, as elaborated below.

Firstly, form the subset $\{2^{1/2}, 2^{1/3}, 2^{1/4}, ..., 2^{1/n}\}$ by taking $m = 2$ and $k = 2$ to n and, note that its cardinal numbers C_1 will be $n - 1$ i.e. $C_1 = n - 1$.

Secondly, form the subset $\{3^{1/2}, 3^{1/3}, 3^{1/4}, ..., 3^{1/n}\}$ by taking $m = 3$ and $k = 2$ to n, and, note that its cardinal number C_2 will also be $n - 1$ i.e. $C_2 = n - 1$.

Thirdly, form the subset $\{4^{1/3}, 4^{1/4}, 4^{1/5}, ..., 4^{1/n}\}$ by taking $m = 4$ and $k = 3$ to n, and, note that its cardinal number C_3 will be $n - 2$, due to the exclusion of $4^{1/2} = 2$ (corresponding to k = 2) i.e. $C_3 = n - 2$.

Proceeding this way, the cardinal number C_{n-1} of the last $(n-1)^{th}$ subset $\{n^{1/2}, n^{1/3}, n^{1/4}, ..., n^{1/n}\}$ will be less than or equal to $(n - 1)$ according as n is of the form m^k or not.

Thus, the required cardinal number C of the finite set of the said irrational numbers formed from 1, 2, 3, ... n, will be less than $(n-1)^2$ i.e. $C < (n-1)^2$.

Accordingly, the cardinal number of the duly modified set **R** will also be $< \lim_{n\to\infty}(n-1)^2 = \lim_{n\to\infty} n^2 = \bar{N}^2$ i.e. infinity of order two. Interestingly, this cardinal number is same as that of duly modified set **Q** but with a difference, as elaborated below.

It is to be noted that the numbers like $4^{1/2} = 2$ or $8^{1/3} = 2$ dropped while forming the above subsets of irrational numbers will be very much less than the numbers like $2/4 = 1/2$ or $2/6 = 1/3$ dropped while forming subsets of fractions due to their more repetitions. Thus, though the cardinal numbers of duly modified **Q** and **R** turn out to be of the same order two, but the difference between them can be visualized, as explained below.

(i) The duly modified set **R** will be more denser than the duly modified set **Q** but much more denser than the set **N**.

(ii) If one has to approach infinity by stepping onto the irrational numbers then it will be slower than stepping onto the fractions but much more slower than stepping onto the natural numbers.

7. Exclusion of Transcendental Numbers from the set R

Transcendental numbers have their genesis in Euler's query in 1748 viz. whether π and e could be termed as algebraic numbers ? But, without waiting for any answer to this question, existence of non-algebraic numbers was taken for granted by terming them as transcendental numbers. However, after about 100 years in 1844, Liouville "constructed" first non-algebraic / transcendental number in the form of an infinite series and later on generalized it to get an infinity of them. Thereafter, in 1873 and 1882 Hermite and Lindomann proved that e and π were transcendental numbers. However, it is still not known whether $e^e, \pi^\pi, \pi^e, \pi + e$ or $e.\pi$ are transcendental numbers.

In 1934 and 1935, Gelfond and Schneider independently proved that all numbers of the form a^b are transcendental if a is algebraic ($\neq 0, 1$) and b is an irrational algebraic number [4]. Many more transcendental numbers have been "constructed", but does any one of them has an absolute value required for its representation on the real number line ?

In fact, many such transcendental numbers will continue to be "constructed" as and when required/desired but all of them, in general, can be classified into following two categories.

(a) Transcendental numbers such as π and e which appear in isolation; such numbers will form a finite set and its cardinality can be determined by counting them but note that it will keep changing with the addition of such numbers.

(b) Transcendental numbers such as Liouville numbers or ones satisfying Gelfond-Schneider conditions; such numbers will form independent infinite sets and their cardinalities can be considered as discussed in sections 5 and 6.

8. Remark

Even if all the transcendental numbers (known as well as unknown) are considered part of the set R, the real numbers cannot be taken to represent a continuum, as explained below.

Real numbers are and will always be expressed in terms of discrete natural numbers and hence there will always be a finite difference between any two of them, however small and close they may be to each other. Interestingly, infinite divisibility is often considered synonymous with continuity but this again is fallacious in view of the above explanation. Furthermore, infinite divisibility and continuity belong to two different worlds - 'abstract' and 'physical', respectively. Lastly, most of the quagmires/paradoxes stem from the non-consideration of the above facts about real numbers.

References

[1] Clawson Calvin C., The Mathematical Traveler : Exploring the Grand History of Numbers, Viva Books Pvt. Ltd., India, (2004); Chap. 8; 135-149.

[2] Ibid., Chap. 10; 189-206.

[3] Ibid., Chap. 12; 223-232.

[4] Ibid., Chap. 10; 183-189.