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A STUDY OF UNIFIED INTEGRALS INVOLVING THE GENERALIZED POLYNOMIAL SET, GENERALIZED LEGENDRE'S ASSOCIATED FUNCTION AND ALEPH (ℵ) FUNCTION WITH APPLICATIONS

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Abstract: In this Paper We evaluate three finite integrals involving the product of generalized sequence of function $S_n^{\mu,\delta,0}$, Generalized Legendre associated function $P_\gamma^{\alpha,\beta}(x)$ and Aleph (ℵ) function. Then we will demonstrate three theorems as an application of our results and future, to use the three results of Orr and Bailey found in the well-known text by Slater [7]. The study also aims to evaluate some new integrals by the applications of these theorems, which retain their general nature and also create an interest by the applications of these theorems.

Keywords: Generalized Sequence of function, Fractional Differential Operator, Aleph (ℵ)-function, General Class of Multivariable Polynomials and Generalized Legendre Associated function.

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1. Introduction

1.1. The Legendre Associated Function:

B. Meulenbeld [6], is defined and represents Generalized Legendre Associated function $P_\gamma^{\alpha,\beta}(x)$ as follows:

$$P_\gamma^{\alpha,\beta}(x) = \frac{(1+x)^{\frac{\beta}{2}}}{(1-x)^{\frac{\alpha}{2}}\Gamma(1-\alpha)}\ {}_2F_1\left\{\begin{matrix}\gamma-\dfrac{\alpha-\beta}{2}+1, -\gamma-\dfrac{\alpha-\beta}{2} & ; \dfrac{1-x}{2}\\ 1-\alpha; & \end{matrix}\right\}, \quad (1.1)$$

where α is non-positive integer and β, γ are unrestricted.
If We Put $\alpha = \beta$ in (1.1), then $P_\gamma^{\alpha,\beta}(x)$ becomes to the Associated Legendre

Function $P_\gamma^\alpha(x)$ [3, p. 999, Eq. (8.704)] and also if we take $\alpha = \beta = 0$ in (1.1), $P_\gamma^{\alpha,\beta}(x)$ reduces into known Legendre Polynomials [9, p.166, Eq.2].

1.2. The generalized Sequence of Functions

The generalized polynomial set is represent by the following Rodrigues type formula [10, p.64, Eq. (2.18)]

$$S_n^{\mu,\delta,\zeta}[x;w,s,q,A,B,m,\xi,l] = (Ax+B)^{-\mu}(1-\tau x^w)^{-\delta/\tau}\times$$
$$\times T_{\xi,l}^{m+n}\left[(Ax+B)^{\mu+qn}(1-\tau x^w)^{\frac{\delta}{\tau}+sn}\right], \tag{1.2}$$

with the Differential Operator

$$T_{k,l} = x^l\left[k + x\frac{d}{dx}\right].$$

The explicit series form this generalized sequence of functions is given by [10, p.71, Eq. (2.3.4)]

$$S_n^{\mu,\delta,\zeta}[x;w,s,q,A,B,m,\xi,l] = B^{qn}x^{l(m+n)}(1-\tau x^l)^{sn}l^{m+n}\times$$
$$\times\sum_{\sigma=0}^{m+n}\sum_{\tau=0}^{\sigma}\sum_{j=0}^{m+n}\sum_{i=0}^{j}\frac{(-1)^j(-j)_i(\mu)_i(-\sigma)_i(-\mu-qn)_i}{\sigma!\tau!i!(1-\mu-j)_i}\times$$
$$\times\left(-\frac{\delta}{\tau}-sn\right)_\sigma\left(\frac{i+\xi+w\tau}{l}\right)_{m+n}\left(\frac{-\tau x^w}{1-\tau x^w}\right)\left(\frac{Ax}{B}\right)^i. \tag{1.3}$$

Some Special Cases of (1.3) are given by Raijada in table form [10]. We shall use the following Special Case

If we put $A = 1, B = 0$ in (1.3) and let $\tau \to 0$ and using the well known results.

$$Lt_{\tau\to 0}(1-\tau x^w)^{\frac{\delta}{\tau}} = \exp(-\delta x^\tau),\ Lt_{|b|\to\infty}(b)_n\left(\frac{z}{b}\right)^n = z^n.$$

Then, we come at the following important Polynomial Set

$$S_n^{\mu,\delta,0}[x] = S_n^{\mu,\delta,0} = [x;w,q,1,0,m,\xi,l] = x^{qn+l(m+n)}l^{m+n}\times$$
$$\times\sum_{\eta=0}^{m+n}\sum_{\tau=0}^{\eta}\frac{(-\eta)\left(\frac{\mu+qn+\xi+\tau w}{l}\right)_{m+n}}{\eta!\tau!}(\delta x^w)^\eta. \tag{1.4}$$

1.3. Aleph (ℵ) function:

Sudland [2] Introduced the Aleph (ℵ) function, however the notation and complete definition is presented here in the following manner in terms and the Mellin-Barnes type integrals

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[z \;\middle|\; \begin{matrix} (a_j,A_j)_{1,n}, [\tau_i(a_{ji},A_{ji})]_{n+1,p_i} \\ \\ (b_j,B_j)_{1,m}, [\tau_i(b_{ji},B_{ji})]_{m+1,q_i} \end{matrix} \right] = \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s) z^{-s} ds, \tag{1.5}$$

for all $z \neq 0$ where $\omega = \sqrt{(-1)}$ and

$$\Omega^{m,n}_{p_i,q_i;\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\prod_{i=1}^{r}\tau_i \prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} - B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji} + A_{ji}s)}. \tag{1.6}$$

The integration path $L = L_{i\gamma\infty}$, $\gamma \in R$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles, assumed to be simple of $\Gamma(1 - a_j - A_j s)$, $j = 1, ..., n$ do not coincide with the pole of $\Gamma(b_j + B_j s)$, $j = 1, ..., m$ the parameter $p_i q_i$ are non-negative integers satisfying: $0 \leq n \leq p_i$, $0 \leq m \leq q_i$, $\tau_i > 0$ for $i = 1, ..., r$. The $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j.b_j, a_{ji}, b_{ji} \in C$. The empty product in (1.6) is interpreted as unity. The existence conditions for the defining integral (1.5) are as following:

$$\phi_l > 0, \ |\arg(z)| < \frac{\pi}{2}\phi_l, \quad l = 1, ..., r \tag{1.7}$$

$$\phi_l \geq 0, \ |\arg(z)| < \frac{\pi}{2}\phi_l, \quad \text{and} \quad R\{\xi_l\} < 0 \tag{1.8}$$

where

$$\phi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l\left(\sum_{j=n+1}^{p_l} A_{j1} + \sum_{j=m+1}^{q_l} B_{jl}\right) \tag{1.9}$$

$$\xi_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l\left(\sum_{j=n+1}^{q_l} b_{j1} - \sum_{j=m+1}^{p_l} a_{jl}\right) + \frac{1}{2}(p_l - q_l), \ l = 1, 2, ..., r. \tag{1.10}$$

For detailed introduction of Aleph (ℵ) function see [2] and [4].
The following Three results will be needed in this paper.
A. By Meulenbeld and Robin [5, p. 343, Eq. (38)]

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_{\gamma}^{\alpha,\beta}(x)dx = \frac{2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}\Gamma\left(1+\rho-\frac{\alpha}{2}\right)\Gamma\left(1+\sigma+\frac{\beta}{2}\right)}{\Gamma(1-\alpha)\Gamma\left(2+\rho+\sigma+\frac{\beta-\alpha}{2}\right)}\times$$

$$\times\ {}_3F_2\left\{\begin{array}{l} \gamma-\dfrac{\alpha-\beta}{2}+1,-\gamma-\dfrac{\alpha-\beta}{2},1+\rho-\dfrac{\alpha}{2} \\ \\ 1-\alpha;2+\rho+\sigma+\dfrac{\beta-\alpha}{2} \end{array};1\right\}, \tag{1.11}$$

where α is a non-positive integer and $Re\left(1+\rho-\frac{\alpha}{2}\right)>0,\quad Re\left(1+\sigma+\frac{\beta}{2}\right)>0.$

B. By Sneddon [8, p. 61, Eq. 2.16 (ii)]

$$\int_0^1 x^{l-1}(1-x)^{m-1}\ {}_pF_q\left\{\begin{array}{l}\alpha_1,...,\alpha_p\\ \\ \beta_1,...,\beta_q\end{array};\frac{1-x}{2}\right\}dx$$

$$=\beta(l,m)\ {}_{p+1}F_{q+1}\left[\begin{array}{l}\alpha_1,...,\alpha_p,m\\ \\ \beta_1,...,\beta_q,l+m\end{array};\frac{1}{2}\right] \tag{1.12}$$

C. By Sneddon [8, p. 61, Eq. 2.16 (iii)]

$$\int_0^1 (1-x^2)^{m-1}\ {}_pF_q\left\{\begin{array}{l}\alpha_1,...,\alpha_p\\ \\ \beta_1,...,\beta_q\end{array};\frac{1-x}{2}\right\}dx$$

$$=\beta\left(\frac{1}{2},m\right)\ {}_{p+1}F_{q+1}\left[\begin{array}{l}\alpha_1,...,\alpha_p,m\\ \\ \beta_1,...,\beta_q,2m\end{array};\frac{1}{2}\right] \tag{1.13}$$

2. Main Results

2.1. First Integral

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_\gamma^{\alpha,\beta}(x)S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times\aleph_{p_i,q_i,\tau_i;r}^{M,N}\left\{ zx^h(1-x)^k \left| \begin{array}{l}(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i}\\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}\end{array}\right.\right\}dx$$

$$=y^{R'+w\eta}l^{m+n}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^\eta\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t!\Gamma(1-\alpha+t)}$$

$$\times\aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2}\left\{ z \left| \begin{array}{l}A^*,(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i}\\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i},B^*\end{array}\right.\right\}dx, \tag{2.1}$$

where

$$A^* = (1-\rho-uR'-uw\eta, h), \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t, k\right),$$

$$B^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-t, (h+k)\right\}$$

and

$$R' = qn + l(m+n).$$

Conditions-

(i). $\alpha > 0, u \geq 0, v \geq 0, h \geq 0, k \geq 0$ (not both zero simultaneously)

(ii). $Re(\rho)+h \min\limits_{1\leq j\leq M}\left[Re\left(\frac{b_j}{\beta_j}\right)\right]+1 > 0$ and $Re(\sigma-\frac{\alpha}{2})+k \min\limits_{1\leq j\leq M}\left[Re\left(\frac{b_j}{\beta_j}\right)\right]+1 > 0$

(iii). Equation (1.7), (1.8), (1.9) and (1.10).

Proof:

To prove the above integral, we first write the Generalized Polynomial set $S_n^{\mu,\delta,\tau}[x]$ in its series form and Generalized Legendre function with the help of (1.4) and (1.1) respectively. The left hand side of (2.1) will be take the following form by expressing Aleph (ℵ) function in terms of contour integral with the help of equation (1.5) and change the order of integration and summation (which is permissible, the condition stated):

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}\frac{(1+x)^{\beta/2}}{(1-x)^{\alpha/2}\Gamma(1-\alpha)}$$

$$\times\ {}_2F_1\left\{\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1, -\gamma-\frac{\alpha-\beta}{2} \\ 1-\alpha\end{matrix}; \frac{1-x}{2}\right\}(yx^u(1-x)^v)^{qn+l(m+n)}l^{m+n}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\left[\delta\left\{yx^u(1-x)^v\right\}^w\right]^\eta$$

$$\times\frac{1}{2\pi i}\int_L \Omega_{p_i,q_i,\tau_i;r}^{M,N}(s)\left\{zx^h(1-x)^k\right\}^{-s}dsdx$$

$$= y^{R'+w\eta}l^{m+n}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^\eta\frac{1}{\Gamma(1-\alpha)}$$

$$\times\frac{1}{2\pi i}\int_L \Omega_{p_i,q_i,\tau_i;r}^{M,N}(s)z^{-s}ds\int_0^1 x^{\rho+uR'+uw\eta-hs-1}$$

$$\times(1-x)^{\sigma-\frac{\alpha}{2}+vR'+vw\eta-ks-1}\,{}_2F_1\left\{\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1,-\gamma-\frac{\alpha-\beta}{2} & ;\dfrac{1-x}{2}\\ 1-\alpha; & \end{matrix}\right\}dx.$$

We can get result after a little Simplification when we evaluate the integral with help of (1.12) and reinterpreting the result thus Obtained in terms of Aleph (ℵ) Function. If we reduce general sequence of Polynomials $S_n^{\mu,\delta}$ to unity and Aleph (ℵ) function to Fox H function, we can come Closer to the known result given by Anandani [1, 9.343, Eq. (2.2)].

2.2. Second Integral

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_{\gamma}^{\alpha,\beta}(x)S_n^{\mu,\delta,0}\{y(1-x)^u(1+x)^v\}$$

$$\times\aleph_{p_i,q_i,\tau_i;r}^{M,N}\left\{ z(1-x)^h(1+x)^k \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i}\\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}\end{matrix}\right\}dx$$

$$=y^{R'+w\eta}l^{m+n}2^{\{R'+w\eta\}(u+v)}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_{\gamma}\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^{\eta}$$

$$\times\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{t!\Gamma(1-\alpha+t)}2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}$$

$$\times\aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2}\left\{ z2^{h+k} \,\middle|\, \begin{matrix}C^*,(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i}\\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i},D^*\end{matrix}\right\}dx, \qquad (2.2)$$

where

$$C^*=\left(\frac{\alpha}{2}-\rho-uR'-uw\eta,h\right),\left(-\sigma-\frac{\beta}{2}-vR'-vw\eta,k\right),$$

$$D^*=\left\{-\rho-\sigma-t-\frac{\beta-\alpha}{2}-(u+v)\{R'+w\eta\}-1,(h+k)\right\}$$

and

$$R'=qn+l(m+n).$$

Conditions-

(i). $\alpha>0,u\geq 0,v\geq 0,h\geq 0,k\geq 0$ (not both zero simultaneously)

(ii). $Re(\rho)+h+\min\limits_{1\leq j\leq M}\left[Re\left(\dfrac{b_j}{\beta_j}\right)\right]+1>0,$

(iii). $Re(\sigma - \frac{\alpha}{2}) + k + \min_{1\leq j\leq M}\left[Re\left(\frac{b_j}{\beta_j}\right)\right] + 1 > 0$

Proof:

To establish the result (2.2) we express Aleph function $\aleph^{M,N}_{p_i,q_i;\tau_i,r}$ in term of Mellin Barnes contour integral with help of (1.5), (1.1) and (1.4). Now changing the order of summation and integration (which is permissible under the condition stated) and evaluate the integral, using (1.11). We get the right hand side of integral (2.2) after a little simplification.

$$= y^{R'+wn} l^{m+n} \sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta \left\{\frac{1}{2\pi i}\int_L \Omega^{M,N}_{p_i,q_i,\tau_i;r}(s) z^{-s} ds\right\}$$

$$\times \frac{1}{\Gamma(1-\alpha)}\, {}_3F_2 \left\{ \begin{matrix} \gamma - \frac{\alpha-\beta}{2} + 1, -\gamma - \frac{\alpha-\beta}{2}, 1 + \rho + uR' + uw\eta - hs - \frac{\alpha}{2} \\ 1-\alpha; \left(2 + \rho + \sigma + \frac{\beta-\alpha}{2} + (u+v)\left\{R' + w\eta - s(h+k)\right\}\right) \end{matrix} ; 1 \right\}$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta \times$$

$$\times 2^{(R'+w\eta)(u+v)} \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2} + 1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{t!\Gamma(1-\alpha+t)} 2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1} \times$$

$$\times \aleph^{M,N+2}_{p_i+2,q_i+1,\tau_i;r} \left\{ z2^{h+k} \left| \begin{matrix} C^*, (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}, D^* \end{matrix} \right. \right\},$$

where

$$C^* = \left(\frac{\alpha}{2} - \rho - uR' - uw\eta, h\right), \left(-\sigma - \frac{\beta}{2} - vR' - vw\eta, k\right),$$

$$D^* = \left\{-\rho - \sigma - t - \frac{\beta-\alpha}{2} - (u+v)\{R' + w\eta\} - 1, (h+k)\right\}$$

2.3. Third Integral

$$\int_0^1 (1-x)^{\alpha/2}(1+x)^{-\beta/2}(1-x^2)^{\rho-1} P^{\alpha,\beta}_\gamma(x) S^{\mu,\delta,0}_n\{y(1-x^2)^k\}$$

$$\times \aleph^{M,N}_{p_i,q_i,\tau_i;r} \left\{ z(1-x^2)^h \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta} l^{m+n}\sqrt{(\pi)}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^\eta$$

$$\times\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^{2t}t!\Gamma(1-\alpha+t)}$$

$$\times\aleph^{M,N+2}_{p_i+2,q_i+2,\tau_i;r}\left\{ z \,\middle|\, \begin{matrix} E^*,(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i},F^* \end{matrix}\right\} dx, \qquad (2.3)$$

where

$$E^* = (1-\rho-kR'-kw\eta,h),(1-\rho-kR'-kw\eta-t,h),$$

$$F^* = \left\{1-\rho-kR'-kw\eta-\frac{t}{2},h\right\},\left(1-\rho-kR'-kw\eta-\frac{t}{2}-\frac{1}{2},h\right)$$

The integral is valid under the following conditions:

(i). $\alpha > 0; h \geq 0, k \geq 0$ (not both zero simultaneously)

(ii). $Re(\rho)+h+\min_{1\leq j\leq M}\left[Re\left(\frac{b_j}{\beta_j}\right)\right]+1>0,$

Proof:

Firstly we express the Generalized Legendre function in terms of ${}_2F_1$ with help of (1.1). Generalized Polynomial set $S_n^{\mu,\delta,0}[x]$ in its series form using (1.4) and, $\aleph^{M,N}_{p_i,q_i;\tau_i,r}$ function in terms of Mellin Barnes contour integral with help of (1.5) in the left hand side of (2.3). Now changing the order of summation and integration (which is permissible under the condition stated) we get the following form of integral:

$$= y^{R'+wn} l^{m+n}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^\eta\frac{1}{2\pi i}\int_L\Omega^{M,N}_{p_i,q_i,\tau_i;r}(s)z^{-s}ds$$

$$\times\frac{1}{\Gamma(1-\alpha)}\beta\left\{\frac{1}{2},\rho+kR'+kw\eta-hs\right\}$$

$$\times\ {}_3F_2\left\{\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1,-\gamma-\frac{\alpha-\beta}{2},\rho+kR'+kw\eta-hs \\ \\ 1-\alpha;2\{\rho+kR'+kw\eta-hs\}\end{matrix};\frac{1}{2}\right\}dx$$

Now interpreting the resulting expression in terms of Aleph ($\aleph$) function and applying well known result (1.13) and duplication formula $\Gamma\left(\frac{1}{2}\right)\Gamma(2n)=2^{2n-1}\ \Gamma(n)$ $\Gamma\left(n+\frac{1}{2}\right)$. We came at the desired result (2.3).

Theorem 2.1.

If $(1-x)^{a+b-c}\ {}_2F_1[2a,2b;2c;x]=\sum_{n'=0}^{\infty}a_{n'}x^{n'}$

Then

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_{\gamma}^{\alpha,\beta}(x)S_{\eta}^{\mu,\delta,0}\left\{yx^u(1-x)^v\right\}$$

$$\times\ {}_2F_1\left[a,b;c+\frac{1}{2};x\right]\ {}_2F_1\left[c-a,c-b;c+\frac{1}{2};x\right]$$

$$\times\aleph_{p_i,q_i,\tau_i;r}^{M,N}\left\{\ zx^h(1-x)^k\ \middle|\ \begin{matrix}(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i}\\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}\end{matrix}\right\}dx$$

$$=y^{R'+w\eta}l^{m+n}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_{\gamma}\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^{\eta}$$

$$\times\sum_{n'=0}^{\infty}\frac{(c)_{n'}}{\left(c+\frac{1}{2}\right)_{n'}}a^{n'}\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t!\Gamma(1-\alpha+t)}$$

$$\times\aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2}\left\{\ z\ \middle|\ \begin{matrix}G^*,(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i}\\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i},H^*\end{matrix}\right\}dx, \tag{2.4}$$

where

$$G^*=(1-\rho-uR'-uw\eta-n',h),\left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right)$$

$$H^*=\left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-n'-t,(h+k)\right\}$$

The conditions of validity given in Integral first are satisfied.

Proof. To prove the theorem, we consider the following result [Slater, L.J.;7, p.75]

$${}_2F_1\left[a,b;c+\frac{1}{2};z\right]\ {}_2F_1\left[c-a,c-b;c+\frac{1}{2};z\right]=\sum_{n'=0}^{\infty}\frac{(c)_{n'}}{\left(c+\frac{1}{2}\right)_{n'}}a^{n'}z^{n'} \tag{2.5}$$

Now, we use an operation (say $\bigwedge$) i.e. multiplication by the term

$$\bigwedge=x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_{\gamma}^{\alpha,\beta}(x)S_{\eta}^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times \aleph^{M,N}_{p_i,q_i,\tau_i;r}\left\{ zx^h(1-x)^k \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{matrix} \right\}$$

Subsequent integration with respect to x from $x = 0$ to 1.
On doing the operation (say $\bigwedge$), on both sides of the equation (2.5) and interchanging the order of integration and summation in the right hand side, we get the following result.

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_\gamma^{\alpha,\beta}(x)S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times\ {}_2F_1\left[a,b;c+\frac{1}{2};x\right]\ {}_2F_1\left[c-a,c-b;c+\frac{1}{2};x\right]$$

$$\times \aleph^{M,N}_{p_i,q_i,\tau_i;r}\left\{ zx^h(1-x)^k \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{matrix} \right\} dx$$

$$= \sum_{n'=0}^{\infty}\frac{(c)_{n'}}{\left(c+\frac{1}{2}\right)}a^{n'}\int_0^1 x^{\rho+n'-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_\gamma^{\alpha,\beta}(x)S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times \aleph^{M,N}_{p_i,q_i,\tau_i;r}\left\{ zx^h(1-x)^k \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{matrix} \right\} \tag{2.6}$$

Finally, we easily reach at the required result by evaluating the integral on the right hand side with help of same procedure as given in the Theorem 2.1.

Theorem 2.2.
If

$${}_2F_1[a,b;c;x]\ {}_2F_1[a,b;d;x] = \sum_{n'=0}^{\infty} c_{n'}x^{n'} \tag{2.7}$$

Then

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}$$

$$\times\ {}_4F_3\left[a,b;\frac{c}{2}+\frac{d}{2},\frac{c}{2}+\frac{b}{2}-\frac{1}{2};a+b,c,d;4x(1-x)\right]\times P_\gamma^{\alpha,\beta}(x)S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times \aleph^{M,N}_{p_i,q_i,\tau_i;r}\left\{ zx^h(1-x)^k \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{matrix} \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(c+d-1)_{n'}}{(a+b)_{n'}} (c)_{n'} \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2} + 1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2} \left\{ z \left| \begin{array}{c} I^*, (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}, J^* \end{array} \right. \right\}, \qquad (2.8)$$

where

$$I^* = (1-\rho-uR'-uw\eta-n',h)\,, \ \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right)$$

$$J^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-n'-t,(h+k)\right\}$$

The conditions of validity given in Integral first are satisfied.

Proof. We have [Slater, L.J.; 7, p.79]

$$ {}_4F_3\left[a,b;\frac{c}{2}+\frac{d}{2},\frac{c}{2}+\frac{b}{2}-\frac{1}{2};a+b,c,d;4x(1-x)\right] = \sum_{n'=0}^{\infty} \frac{(c+d-1)_{n'}}{(a+b)_{n'}} (c)_{n'} x^{n'} \qquad (2.9)$$

where $c_{n'}$ is given by (2.7). We reach at the desired result (2.8) by performing the same procedure as given in the Theorem 2.1.

Theorem 2.3.

If

$$ {}_2F_1[a,b;c;x]\ {}_2F_1[a,b;d;x] = \sum_{n'=0}^{\infty} c_{n'} x^{n'} \qquad (2.10)$$

Then

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}$$

$$\times\ {}_4F_3\left[a,b,d,c-a;\frac{b}{2}+\frac{d}{2},\frac{b}{2}+\frac{d}{2}+\frac{1}{2},c;\frac{-x^2}{4(1-x)}\right] \times P_\gamma^{\alpha,\beta}(x) S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times \aleph_{p_i,q_i,\tau_i;r}^{M,N} \left\{ zx^h(1-x)^k \left| \begin{array}{c} (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{array} \right. \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(c)_{n'}(d)_{n'}}{(d+b)_{n'}} \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2} \left\{ z \;\middle|\; \begin{matrix} I^*, (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}, J^* \end{matrix} \right\}, \tag{2.11}$$

where

$$I^* = (1-\rho-uR'-uw\eta-n', h), \ \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t, k\right)$$

$$J^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-n'-t, (h+k)\right\}$$

Where, the Conditions stated in (2.1) are true.

Proof. We have [7]

$$(1-z)^{-a}\ {}_4F_3\left[a, b, d, c-a; \frac{b}{2}+\frac{d}{2}, \frac{b}{2}+\frac{d}{2}+\frac{1}{2}, c; \frac{-z^2}{4(1-z)}\right] = \sum_{n'=0}^{\infty} \frac{(c)_{n'}(d)_{n'}}{(d+b)_{n'}} z^{n'} \tag{2.12}$$

We easily arrive at the right hand side of (2.11) by proceeding on similar lines to those of theorem 2.1.

3. Applications and Special Case

(I) Taking $c = a$ in our theorem 2.1, then value of $(a)_n$ in (2.5) come out to be equal to $(b)_n$ and the results yields the following interesting integral:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x) S_\eta^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$${}_2F_1\left[a, b; c+\frac{1}{2}; x\right] \aleph_{p_i,q_i,\tau_i;r}^{M,N} \left\{ zx^h(1-x)^k \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{matrix} \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(a)_{n'}}{\left(a+\frac{1}{2}\right)_{n'} n'!} a^{n'} \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2} \left\{ z \;\middle|\; \begin{matrix} G^*, (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}, H^* \end{matrix} \right\}, \qquad (3.1)$$

where

$$G^* = (1-\rho-uR'-uw\eta-n',h), \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right)$$

$$H^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-n'-t,(h+k)\right\}$$

Again putting $b = \frac{a}{2}+1$ and $a = -e$ (a non negative integral) in (3.1).

We have

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x) S_\eta^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$${}_1F_0[-e;-;x]\, \aleph_{p_i,q_i,\tau_i;r}^{M,N} \left\{ zx^h(1-x)^k \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{matrix} \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(-e)_{n'}}{n'!} \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2} \left\{ z \;\middle|\; \begin{matrix} G^*, (a_j,\alpha_j)_{1,N}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M}, ..., [\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}, H^* \end{matrix} \right\}, \qquad (3.2)$$

where

$$G^* = (1-\rho-uR'-uw\eta-n',h), \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right)$$

$$H^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-n'-t,(h+k)\right\}$$

(II) We easily achieve the following result on Putting $b = c = d$ in theorem 2.2

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}\, {}_2F_1\left[a, c-\frac{1}{2}; a+c; 4x(1-x)\right] P_\gamma^{\alpha,\beta}(x)$$

$$S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}\aleph_{p_i,q_i,\tau_i;r}^{M,N}\left\{ zx^h(1-x)^k \;\middle|\; \begin{array}{c} (a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{array} \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(2c-1)_{n'}}{(a+c)_{n'}} \frac{(2a)_{n'}}{n'!} \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2}\left\{ z \;\middle|\; \begin{array}{c} I^*,(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}, J^* \end{array} \right\}, \tag{3.3}$$

where

$$I^* = (1-\rho-uR'-uw\eta-n',h), \;\; \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right)$$

$$J^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-n'-t,(h+k)\right\}$$

Further, if we Substitute $a=-e$ in (3.3), it reduces to the following interesting integral

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}\, {}_2F_1\left[e,c-\frac{1}{2};a-e;4x(1-x)\right] P_\gamma^{\alpha,\beta}(x)$$

$$S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}\aleph_{p_i,q_i,\tau_i;r}^{M,N}\left\{ zx^h(1-x)^k \;\middle|\; \begin{array}{c} (a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i} \end{array} \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(2c-1)_{n'}}{(a-e)_{n'}} \frac{(-2e)_{n'}}{n'!} \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2}\left\{ z \;\middle|\; \begin{array}{c} I^*,(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}, J^* \end{array} \right\}, \tag{3.4}$$

where

$$I^* = (1-\rho-uR'-uw\eta-n',h)\,,\ \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right)$$

$$J^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-n'-t,(h+k)\right\}$$

(III) We reach at the following integral If we put $b=c=d$ in theorem 2.3

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}\ {}_2F_1\left[a,c-a;c+\frac{1}{2};\frac{-x^2}{4(1-x)}\right]P_\gamma^{\alpha,\beta}(x)$$

$$S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}\aleph_{p_i,q_i,\tau_i;r}^{M,N}\left\{\ zx^h(1-x)^k\ \middle|\ \begin{matrix}(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i}\\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i}\end{matrix}\right\}dx$$

$$= y^{R'+w\eta}l^{m+n}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^\eta$$

$$\times\sum_{n'=0}^{\infty}\frac{(c)_{n'}(2a)_{n'}}{(2c)_{n'}\,n'!}\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t!\Gamma(1-\alpha+t)}$$

$$\times\aleph_{p_i+2,q_i+1,\tau_i;r}^{M,N+2}\left\{\ z\ \middle|\ \begin{matrix}I^*,(a_j,\alpha_j)_{1,N},...,[\tau_i(a_{ji},\alpha_{ji})]_{N+1,p_i}\\ \\ (b_j,\beta_j)_{1,M},...,[\tau_i(b_{ji},\beta_{ji})]_{M+1,q_i},J^*\end{matrix}\right\},\tag{3.5}$$

where

$$I^* = (1-\rho-uR'-uw\eta-n',h)\,,\ \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right)$$

$$J^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-n'-t,(h+k)\right\}$$

(IV) Aleph function reduce to I-function if we choose $\tau_i = 1$.
(i) By Integral First

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_\gamma^{\alpha,\beta}(x)S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times I_{p_i,q_i;r}^{M,N}\left\{\ zx^h(1-x)^k\ \middle|\ \begin{matrix}(a_j,\alpha_j)_{1,N},...,(a_{ji},\alpha_{ji})_{N+1,p_i}\\ \\ (b_j,\beta_j)_{1,M},...,(b_{ji},\beta_{ji})_{M+1,q_i}\end{matrix}\right\}dx$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2} + 1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\times I^{M,N+2}_{p_i+2,q_i+1;r} \left\{ z \left| \begin{matrix} A^*, (a_j, \alpha_j)_{1,N}, ..., (a_{ji}, \alpha_{ji})_{N+1,p_i} \\ \\ (b_j, \beta_j)_{1,M}, ..., (b_{ji}, \beta_{ji})_{M+1,q_i}, B^* \end{matrix} \right. \right\} dx, \tag{3.6}$$

where

$$A^* = (1-\rho-uR'-uw\eta, h), \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t, k\right),$$

$$B^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-t, (h+k)\right\}$$

(ii) By Integral Second

$$\int_{-1}^{1} (1-x)^\rho (1+x)^\sigma P_\gamma^{\alpha,\beta}(x) S_n^{\mu,\delta,0}\{y(1-x)^u(1+x)^v\}$$

$$\times I^{M,N}_{p_i,q_i;r} \left\{ z(1-x)^h(1+x)^k \left| \begin{matrix} (a_j, \alpha_j)_{1,N}, ..., (a_{ji}, \alpha_{ji})_{N+1,p_i} \\ \\ (b_j, \beta_j)_{1,M}, ..., (b_{ji}, \beta_{ji})_{M+1,q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} 2^{\{R'+w\eta\}(u+v)} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2} + 1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{t! \Gamma(1-\alpha+t)} 2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}$$

$$\times I^{M,N+2}_{p_i+2,q_i+1;r} \left\{ z2^{h+k} \left| \begin{matrix} C^*, (a_j, \alpha_j)_{1,N}, ..., (a_{ji}, \alpha_{ji})_{N+1,p_i} \\ \\ (b_j, \beta_j)_{1,M}, ..., (b_{ji}, \beta_{ji})_{M+1,q_i}, D^* \end{matrix} \right. \right\} dx, \tag{3.7}$$

where

$$C^* = \left(\frac{\alpha}{2}-\rho-uR'-uw\eta, h\right), \left(-\sigma-\frac{\beta}{2}-vR'-vw\eta, k\right),$$

$$D^* = \left\{-\rho-\sigma-t-\frac{\beta-\alpha}{2}-(u+v)\{R'+w\eta\}-1, (h+k)\right\}$$

(iii) By Integral Third

$$\int_{0}^{1} (1-x)^{\alpha/2} (1+x)^{-\beta/2} (1-x^2)^{\rho-1} P_\gamma^{\alpha,\beta}(x) S_n^{\mu,\delta,0}\{y(1-x^2)^k\}$$

$$\times I^{M,N}_{p_i,q_i;r}\left\{ z(1-x^2)^h \;\middle|\; \begin{array}{c} (a_j,\alpha_j)_{1,N},...,(a_{ji},\alpha_{ji})_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M},...,(b_{ji},\beta_{ji})_{M+1,q_i} \end{array} \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sqrt{(\pi)} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^{2t} t! \Gamma(1-\alpha+t)}$$

$$\times I^{M,N+2}_{p_i+2,q_i+2;r}\left\{ z \;\middle|\; \begin{array}{c} E^*,(a_j,\alpha_j)_{1,N},...,(a_{ji},\alpha_{ji})_{N+1,p_i} \\ \\ (b_j,\beta_j)_{1,M},...,(b_{ji},\beta_{ji})_{M+1,q_i},F^* \end{array} \right\} dx, \tag{3.8}$$

where

$$E^* = (1-\rho-kR'-kw\eta,h),(1-\rho-kR'-kw\eta-t,h),$$

$$F^* = \left\{1-\rho-kR'-kw\eta-\frac{t}{2},h\right\}, \left(1-\rho-kR'-kw\eta-\frac{t}{2}-\frac{1}{2},h\right)$$

(V) Aleph function reduce to H-function if we choose $\tau_1 = 1, r = 1$
(i) By Integral First

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P^{\alpha,\beta}_\gamma(x) S^{\mu,\delta,0}_n\{yx^u(1-x)^v\}$$

$$\times H^{M,N}_{p,q}\left\{ zx^h(1-x)^k \;\middle|\; \begin{array}{c} (a_1,\alpha_1),...,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),...,(b_q,\beta_q) \end{array} \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!} \delta^\eta \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t t! \Gamma(1-\alpha+t)}$$

$$\times H^{M,N+2}_{p+2,q+1}\left\{ z \;\middle|\; \begin{array}{c} A^*,(a_1,\alpha_1),...,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),...,(b_q,\beta_q),B^* \end{array} \right\} dx, \tag{3.9}$$

where

$$A^* = (1-\rho-uR'-uw\eta,h), \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right),$$

$$B^* = \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\{R'+w\eta\}-t,(h+k)\right\}$$

(ii) By Integral Second

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_{\gamma}^{\alpha,\beta}(x)S_{n}^{\mu,\delta,0}\{y(1-x)^{u}(1+x)^{v}\}$$

$$\times H_{p,q}^{M,N}\left\{\begin{array}{c|c} z(1-x)^{h}(1+x)^{k} & \begin{array}{c}(a_1,\alpha_1),...,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),...,(b_q,\beta_q)\end{array}\end{array}\right\}dx$$

$$=y^{R'+w\eta}l^{m+n}2^{\{R'+w\eta\}(u+v)}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_{\gamma}\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^{\eta}$$

$$\times\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_{t}\left(-\gamma-\frac{\alpha-\beta}{2}\right)_{t}}{t!\Gamma(1-\alpha+t)}2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}$$

$$\times H_{p+2,q+1;r}^{M,N+2}\left\{\begin{array}{c|c} z2^{h+k} & \begin{array}{c}C^{*},(a_1,\alpha_1),...,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),...,(b_q,\beta_q),D^{*}\end{array}\end{array}\right\}dx, \tag{3.10}$$

where

$$C^{*}=\left(\frac{\alpha}{2}-\rho-uR'-uw\eta,h\right),\left(-\sigma-\frac{\beta}{2}-vR'-vw\eta,k\right),$$

$$D^{*}=\left\{-\rho-\sigma-t-\frac{\beta-\alpha}{2}-(u+v)\{R'+w\eta\}-1,(h+k)\right\}$$

(iii) By Integral Third

$$\int_{0}^{1}(1-x)^{\alpha/2}(1+x)^{-\beta/2}(1-x^{2})^{\rho-1}P_{\gamma}^{\alpha,\beta}(x)S_{n}^{\mu,\delta,0}\{y(1-x^{2})^{k}\}$$

$$\times H_{p,q}^{M,N}\left\{\begin{array}{c|c} z(1-x^{2})^{h} & \begin{array}{c}(a_1,\alpha_1),...,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),...,(b_q,\beta_q)\end{array}\end{array}\right\}dx$$

$$=y^{R'+w\eta}l^{m+n}\sqrt{(\pi)}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_{\gamma}\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\eta!}\delta^{\eta}$$

$$\times\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_{t}\left(-\gamma-\frac{\alpha-\beta}{2}\right)_{t}}{2^{2t}t!\Gamma(1-\alpha+t)}$$

$$\times H^{M,N+2}_{p+2,q+2}\left\{ z \;\middle|\; \begin{matrix} E^*, (a_1,\alpha_1),...,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),...,(b_q,\beta_q), F^* \end{matrix} \right\} dx, \tag{3.11}$$

where

$$E^* = (1-\rho-kR'-kw\eta, h), (1-\rho-kR'-kw\eta-t, h),$$

$$F^* = \left\{1-\rho-kR'-kw\eta-\frac{t}{2}, h\right\}, \left(1-\rho-kR'-kw\eta-\frac{t}{2}-\frac{1}{2}, h\right)$$

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