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THE TOTAL EDGE GEODETIC DOMINATION NUMBER OF A GRAPH

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Abstract: In this paper the concept of total edge geodetic domination number of a graph is introduced. A set of vertices S of a graph G is called a total edge geodetic set if S is an edge geodetic set and its induced subgraph has no isolated vertices. The minimum cardinality of all total edge geodetic sets of G is called the total edge geodetic number and is denoted by $g_{et}(G)$. A total edge geodetic dominating set is an edge geodetic dominating set and its induced subgraph has no isolated vertices. The minimum cardinality of all such total edge geodetic dominating sets of G is called the total edge geodetic domination number and is denoted by $\gamma_{get}(G)$. It is shown that for every pair of integers a, b and c such that $2 \leq a \leq b \leq c$, there exist a connected graph G of order p with $g_e(G) = a, \gamma_{ge}(G) = b$ and $\gamma_{get}(G) = c$. Also, for any positive integers m, p with $3 \leq m \leq p$ then there is a connected graph G of order p such that $\gamma_{get}(G) = m$.

Keywords: Edge Geodetic set, Edge geodetic number, Edge geodetic dominating set, Edge geodetic domination number, Total Edge geodetic dominating set, Total Edge geodetic domination number.

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1. Introduction

By a Graph $G = (V, E)$, we mean a simple graph of order at least two. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology, (see [3], [6]). The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighbourhood of a vertex v is the set $N[v] = N(v) \cup N\{v\}$. A vertex v is an extreme vertex if the sub graph induced by its neighbours is complete. A vertex v is a semi-extreme vertex of G if the sub graph induced by its neighbours has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex.(see [12]).

For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A geodetic set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets. An edge geodetic set of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_e(G)$ of G is the minimum order of its edge geodetic sets.(see [4], [7]).

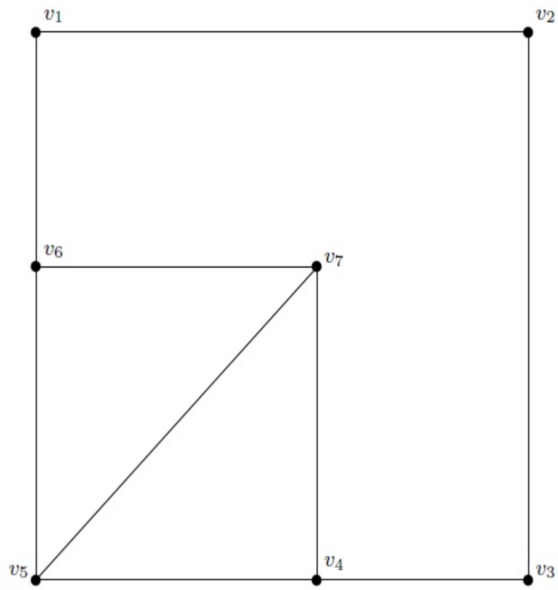
A dominating set in a graph G is a subset of vertices of G such that every vertex outside the subset has neighbour in it. The size of a minimum dominating set in a graph G is called the domination number of G and is denoted by $\gamma(G)$. A geodetic dominating set of G is a subset of $V(G)$ which is both geodetic and dominating set of G. The minimum cardinality of a geodetic dominating set is denoted by $\gamma_g(G)$.(see [5], [8], [9], [10], [11], [13], [14]).

An edge geodetic dominating set of G is a subset of $V(G)$ which is both edge geodetic and dominating set of G. The minimum cardinality of an edge geodetic dominating set is denoted by $\gamma_{ge}(G)$. An edge dominating set S is said to be a total edge dominating set if its induced subgraph has no isolated vertices.

2. Total Edge Geodetic Domination Number of a Graph

Definition 1. A total edge geodetic dominating set of a graph G is an edge geodetic dominating set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality among all the total edge geodetic dominating set of G is called the total edge geodetic domination number and is denoted by $\gamma_{get}(G)$.

Example 2. *Consider the graph given in Figure 01. Here* $S_1 = \{v_1, v_4, v_5, v_6\}$ *is a minimum total geodetic dominating set and* $S_2 = \{v_1, v_4, v_5, v_6, v_7\}$ *is a minimum total edge geodetic dominating set. Therefore* $\gamma_{gt}(G) = 4$ *and* $\gamma_{get}(G) = 5$.



G Figure 01

Definition 3. A vertex v of a connected graph G is called a support vertex of G if it is adjacent to an end vertex of G.

Theorem 4. *Each extreme vertex belongs to every total edge geodetic dominating set.*

Proof. Since each extreme vertex belongs to every edge geodetic dominating set, these extreme vertices also belong to every total edge geodetic dominating set.

Remark 5. *Every total edge geodetic dominating set is a total geodetic dominating set, whereas the converse is not true. In the Figure 01, the set $\{v_1, v_4, v_5, v_6\}$ is a minimum total geodetic dominating set but it is not a total edge geodetic dominating set.*

Theorem 6. *If a graph G of order p has no isolated vertices, then $\gamma_{get}(G) \leq p - \delta(G) + 1$.*

Theorem 7. *Each extreme vertex and each support vertex of a connected graph G belong to every total edge geodetic dominating set of G. If the set of all extreme vertices and support vertices form a total edge geodetic dominating set, then it is the unique minimum total edge geodetic dominating set of G.*

Proof. Since every total edge geodetic dominating set is an edge geodetic dominating set , by Theorem 4, each extreme vertex belongs to every total edge geodetic dominating set. Since a total edge geodetic dominating set has no isolated vertices, it follows that each support vertex of G also belongs to every total edge geodetic dominating set.

Corollary 8. *For the complete graph $K_p(p \geq 2), \gamma_{get}(G) = p$.*

Theorem 9. *Let G be a connected graph with cut-vertices and let S be a total edge geodetic dominating set of G. If u is a cut-vertex of G then every component of $G - u$ contains an element of S.*

Proof. Since every total edge geodetic dominating set is an edge geodetic dominating set, the result follows.

Theorem 10. *For any connected graph G,*
$2 \leq \gamma_{gt}(G) \leq \gamma_{get}(G) \leq p$.

Proof. Any total geodetic dominating set needs at least two vertices and so $\gamma_{gt}(G) \geq 2$. Since every total edge geodetic dominating set is a total geodetic dominating set of G, $\gamma_{gt}(G) \leq \gamma_{get}(G)$. Since $V(G)$ is a connected set of G, it is clear that $\gamma_{get}(G) \leq p$. Hence $2 \leq \gamma_{gt}(G) \leq \gamma_{get}(G) \leq p$.

Remark 11. *The bounds in Theorem 10 are sharp. In the Example 2, $p = 7, \gamma_{gt}(G) = 4, \gamma_{get}(G) = 5$.*

Corollary 12. *Let G be a connected graph. If $\gamma_{get}(G) = 2$ then $\gamma_{ge}(G) = 2$.*

Remark 13. *For the complete graph $G = K_2, \gamma_{get}(G) = 2$ and for the complete graph $K_p, \gamma_{get}(G) = p$ so that the total edge geodetic domination number of a graph attains its least value 2 and largest value p.*
Also we notice that for any cycle of order at least 4, the edge geodetic domination number is 2, whereas the total edge geodetic domination number is 3. This shows that the converse of the Corollary 12 need not be true.
The following theorem characterizes graphs for which $\gamma_{get}(G) = 2$.

Theorem 14. *For any connected graph $G, \gamma_{get}(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $\gamma_{get}(G) = 2$. Conversely, let $\gamma_{get}(G) = 2$. Let $S = \{u, v\}$ be a minimum total edge geodetic dominating set of G. Then uv is an edge. It is clear that a vertex different from u and v cannot lie on a u - v geodesic and so $G = K_2$.

Theorem 15. *Let G be a connected graph with at least 2 vertices. Then $\gamma_{get}(G) \leq 2\gamma_g(G)$.*

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a minimum edge geodetic dominating set of G.

Let $y_i \in N(x_i)$ for $i = 1, 2, \dots, k$ and $T = \{y_1, y_2, \dots, y_k\}$. Then $S \cup T$ is a total edge geodetic dominating set of G so that $\gamma_{get}(G) \leq |S \cup T| \leq 2\ \gamma_g(G)$.

Theorem 16. *For the complete bipartite graph $G = K_{m,n}$,*

$$\gamma_{get}(G) = \begin{cases} 2 & \text{if} \quad m = n = 1, \\ m + n & \text{if} \quad m = 1 \text{ and } n \geq 2, \\ min\,\{m, n\} + 1 & \text{if} \quad m, n \geq 2 \end{cases}$$

Theorem 17. *Let G be a connected graph of order $p \geq 3$. Then $\gamma_{get}(G) = 3$ if and only if $G = K_3$ or $G = \overline{K_2} + H$, where H is a graph of order p - 2.*

Proof. First, suppose that $G = K_3$. Then by Corollary 8, $\gamma_{get}(G) = 3$. Next, suppose that $G = \overline{K_2} + H$, where H is a graph of order p 2. Let $V(\overline{K_2}) = \{u_1, u_2\}$. Then for any vertex v of H, the set $S = \{v, u_1, u_2\}$ is a minimum total edge geodetic dominating set of G and so $\gamma_{get}(G) = 3$. Conversely, let $\gamma_{get}(G) = 3$. Let $S_1 = \{v_1, v_2, v_3\}$ be a minimum total edge geodetic dominating set of G. Then the subgraph induced by S_1 contains at least two edges, say v_1v_2 and v_2v_3. If v_1v_3 is an edge, then it follows that $G = K_3$. If v_1v_3 is not an edge, then each vertex $v \notin S_1$ must lie on a geodesic joining v_1 and v_3 so that this geodesic is of length 2. It follows that v is adjacent to both v_1 and v_3. Hence $G = \overline{K_2} + H$, where $V(\overline{K_2}) = \{v_1, v_3\}$ and H is a graph of order $p - 2$.

Corollary 18. *For any connected graph G with k extreme vertices,* $\max\{2, k\} \leq \gamma_{get}(G) \leq p$.

Proof. This follows from Theorems 4 and 7.

Corollary 19. *Let G be a connected graph of order $p \geq 3$. If G contains exactly one universal vertex, then $\gamma_{get}(G) = p - 1$.*

Theorem 20. *Let G be a connected graph of order $p \geq 3$. If G contains a cut-vertex of degree $p - 1$ then $\gamma_{get}(G) = p - 1$.*

Proof. Let v be a cut vertex of G of degree $p - 1$. Then it follows that v is the only vertex of degree $p - 1$. Hence by Corollary 19, $\gamma_{get}(G) = p - 1$.

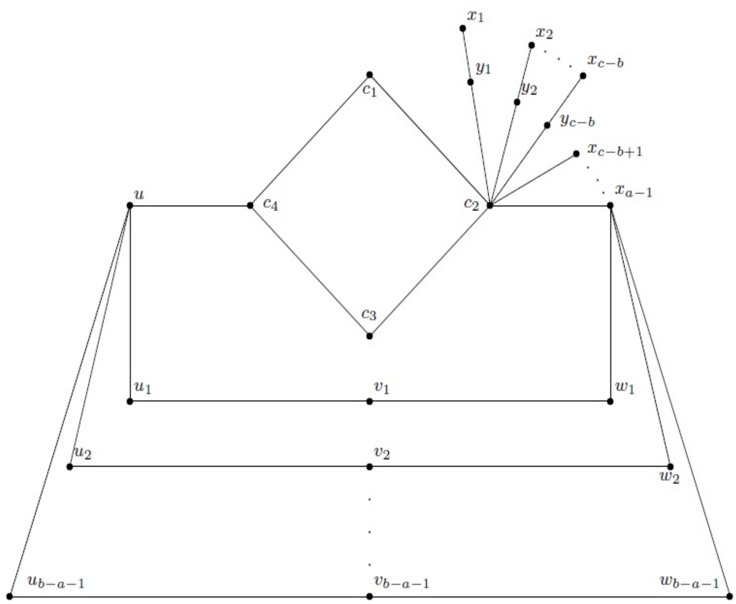
3. Realisation Results

Theorem 21. *For any three integers $2 \leq a \leq b \leq c$, there exists a connected graph G with $g_e(G) = a$, $\gamma_{ge}(G) = b$ and $\gamma_{get}(G) = c$.*

Proof. Let $H_1 : c_1, c_2, c_3, c_4$ be a copy of C_4. Let H_2 be a graph obtained from H_1 by adding a copy of star $K_{1,a-1}$ with leaves $x_1, x_2, \dots, x_{a-1}$ and the support vertex c_2. Let H_3 be the graph obtained from H_1 by subdivide the edges xx_i, where

$1 \leq i \leq c-b$, calling the new vertices $y_1, y_2, \ldots, y_{c-b}$ where x_i is adjacent to y_i. We then introduce three set of $b-a-1$ vertices $u_1, u_2, \ldots, u_{b-a-1}$ and $v_1, v_2, \ldots, v_{b-a-1}$ and $w_1, w_2, \ldots, w_{b-a-1}$.

Also each vertex $u_i\,(1 \leq i \leq b-a-1)$ is adjacent with v_i and each v_i $(1 \leq i \leq b-a-1)$ is adjacent with w_i. Let G be the graph obtained by joining each $u_i\,(1 \leq i \leq b-a-1)$ to a new vertex u and joining each $w_i\,(1 \leq i \leq b-a-1)$ to x_{a-1} as shown in the Figure 02.

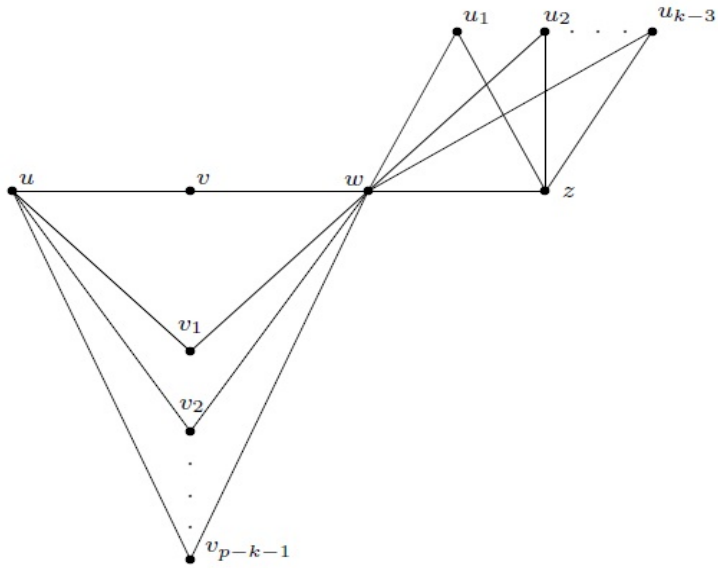


G Figure 02

Let $S_1 = \{x_1, x_2, \ldots, x_{a-1}, u\}$. Then S_1 is a minimum edge geodetic set and therefore $g_e(G) = (a-1)+1 = a$. Let $S_2 = S_1 \cup \{c_2, u_1, u_2, \ldots, u_{b-a-1}\}$. Then S_2 is a minimum edge geodetic dominating set and hence $\gamma_{ge}(G) = a+(b-a-1)+1 = b$. Let $S_3 = S_2 \cup \{y_1, y_2, \ldots, y_{c-b}\}$. Clearly S_3 is a minimum total edge geodetic dominating set and hence $\gamma_{get}(G) = b+c-b = c$.

Theorem 22. *For every pair k, p of integers with $3 \leq k \leq p$, there exists a connected graph of order p such that $\gamma_{get}(G) = k$.*

Proof. Let u, v, w, z be a path on four vertices. Take $k-3$ new vertices $u_1, u_2, \ldots, u_{k-3}$ and join each u_i with w and z. Also take $p-k-1$ new vertices $v_1, v_2, \ldots, v_{p-k-1}$ and join each v_i with the two vertices u and w, we get the connected graph G (see Figure 03). Clearly $|V(G)| = (k-3)+4+(p-k-1) = p$.



G Figure 03

The set of extreme vertices $S_1 = \{u_1, u_2, \ldots, u_{k-3}\}$ is not an edge geodetic dominating set of G. Clearly $S_2 = S_1 \cup \{u, z\}$ is a minimum edge geodetic dominating set of G. Let $S_3 = S_2 \cup \{v\}$. Clearly S_3 is a minimum total edge geodetic dominating set of G and $|S_3| = k - 3 + 3 = k$. Hence $\gamma_{get}(G) = k$.

References

[1] P. Arul Paul Sudhahar, A. Ajitha and A. Subramanian, The Edge Geodetic Domination Number of a Graph, International Journal of Mathematics and its Applications, **4** (2016), 45 - 50.

[2] M. Atici, On The Edge Geodetic Number of a Graph, Int. J. Computer maths **80-7** (2003), 853 - 61.

[3] Buckley and F. Harary, Distance in Graphs, Addison-Wesley, Redwood City, (1990).

[4] G. Chartrand, F. Harary, P. Zhang, On the Geodetic Number of a Graph, Networks, **39** (2002), 1 - 6.

[5] David Amos and Ermelinda Dela Vina, On Total Domination in Graphs, (2012).

[6] Gary Chartrand and P. Zhang, Introduction to Graph Theory, Mac Graw Hill (2015).

[7] Hansberg, A. L. Volkmann, On The Geodetic and Geodetic Domination Numbers of a Graph. Discrete Mathematics, **310(15)** (2015), 2140 46.

[8] T. W. Haynes, S. T. Hedetniemi and P. J Slater, Fundamentals of Domination in Graphs, Marcel Dekker Inc., New York, **208** (1998).

[9] Maryan Atapour and Nasrin Soltankhah, On Total Domination in Graphs, Int J. Contemp. Math. Sciences, **4** (2009), 253 - 57.

[10] Michael A. Henning, A Survey of Selected Recent Results on Total Domination in Graphs, Discrete Mathematics, **309** (2009), 32 - 63.

[11] Robinson Chellathurai S., and Padma Vijaya S., Upper Geodetic Domination Number of a Graph, Int. Journal of Contemporary Math. Sci., **10(1)** (2015), 23 - 36.

[12] Shrinivas S. G. and Vetrivel S., Applications of Graph Theory in Computer Science an Overview, Int. Journal of Eng. Sci. and Tec., **2(9)** (2010), 4610 - 21.

[13] Velammal S. and Arumugam S., The Total Edge Domination in Graphs, Global Journal of Theoretical and Applied Mathematics Sciences, **2** (2012), 79 - 89.

[14] Velammal S., Equality of connected edge domination and total edge domination in graphs, International Journal of Enhanced Research in Science Technology and Engineering, **3** (2014), 198 - 201.