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A STUDY OF QUEUES IN TANDEM RELEVANT TO HOSPITAL SERVICES

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Abstract: Single channel queueing model consisting of two stages of service has been considered. To resolve the system in steady state, matrix method, in view of linear operator, has been applied and some interesting results have been derived.

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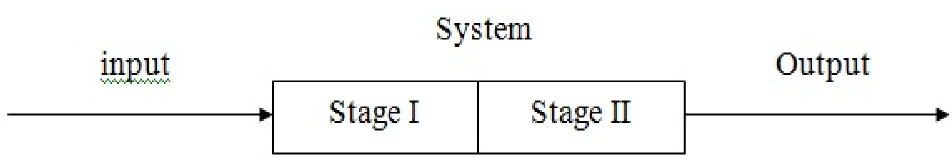
1. Introduction

In hospital service situation exist; patients have to pass successively through several stages to get specialized service of every stage one after another in a series. This situation is known as queue in series or queue in tandem. For example, some patients arriving at the registration counter have to go to, first for a dressings specialist check up and then he goes for some types of immunization service or, dressings for infections etc, done by a different person servicing the patients. There will be two stages of services.

Let us take the simplified one channel queuing model consisting of two stages of service as shown in figure:-



A patient arriving for service has to go through stage I and stage II. The service times at each are exponentially distributed with service rate μ, and arrivals occur according to a Poisson distribution with arrival rate λ. Each stage may be either free or busy. Stage first will be blocked if the patient at this stage completes service before stage second become free. Thus the symbol $o, 1, b$, are used to represent free, busy and blocked states, respectively. The states (i, j), where i represent the state of the stage first and j represent the state of the stage second, are given by, $[i, j] = [(0, 0); (1, 0); (0, 1); (1, 1); (b, 1)]$

If we take $P_{ij}(t)$ as the probability that the system is in state (i, j) at time t. In a recent investigation, Pandey [2] studied a simple model which envisages Poisson input and exponential service time with FCFS queue discipline. The queue physically contain patients (as customers) and doctor behaves as server. The patients are often able to directly estimate the waiting time. But it is not always possible to estimate Pandey [2] obtained the following equations for the single server (doctor) model:

$$\left.\begin{array}{l} P_{00}(t+\Delta t) = P_{00}(t)(1-\lambda\Delta t) + P_{01}(t)\mu\Delta t \\ P_{0,1}(t+\Delta t) = P_{0,1}(t)(1-\mu\Delta t-\lambda\Delta t) + P_{0,1}(t)\mu\Delta t + P_{b1}(t)\mu\Delta t \\ P_{1,0}(t+\Delta t) = P_{00}(t)(\lambda\Delta t) + P_{10}(t)(1-\mu\Delta t) + P_{11}(t)\mu\Delta t \\ P_{11}(t+\Delta t) = P_{01}(t)\lambda\Delta t + P_{11}(t)(1-2\mu\Delta t) \\ P_{b1}(t+\Delta t) = P_{11}(t)\mu\Delta t + P_{b1}(t)(1-\mu\Delta t) \end{array}\right] \tag{1}$$

In steady state these equations can be put as

$$\left.\begin{array}{l} P_{0,1} - pp_{0,0} = 0 \\ P_{1,0} + P_{b,1} - (1+p)P_{0,1} = 0 \\ \rho p_{0,0} + P_{1,1} - P_{1,1} = 0 \\ \rho p_{0,0} + P_{1,1} - P_{1,1} = 0 \\ P_{1,1} - P_{b,1} = 0 \end{array}\right] \tag{2}$$

With the necessary condition

$$P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1} + P_{b,1} = 1 \tag{3}$$

The purpose of the present study is to solve the above system of steady state equations by means of linear operator and matrix method. Moreover, we have also derived expressions for expected number of patients in queue and expected waiting time in queue.

Proposed procedure:

To put our proposed procedure in perspective, we make first an appeal to linear operator [1]

$$D^1 P_{n,i} = P_{n+1,i}$$

Equation (2) can be put as

$$\left.\begin{array}{l} P_{0,1} - \rho P_{0,0} = 0 \\ DP_{0,0} + D^b P_{0,1} - (1+\rho)P_{0,1} = 0 \\ \rho p_{0,0} + DP_{0,1} - DP_{0,0} = 0 \\ \rho p_{0,1} - 2DP_{0,1} = 0 \\ DP_{0,1} - D^b P_{0,1} = 0 \end{array}\right] \tag{4}$$

Equation (5) can be represented in matrix form as follows

$$\begin{bmatrix} -\rho & 1 \\ D & D^b - (1+\rho) \\ \rho - D & D \\ 0 & \rho - 2D \\ 0 & D - D^b \end{bmatrix} \begin{bmatrix} P_{0,0} \\ P_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

After a little simplification, the above equation finally can be put as

$$\begin{bmatrix} 1 & -\rho \\ 0 & D - \rho[(1+\rho) - D^b] \\ 0 & \rho - D + \rho D \\ 0 & -\rho(2D - \rho) \\ 0 & -\rho(D^b - D) \end{bmatrix} \begin{bmatrix} P_{0,0} \\ P_{0,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

Expressing (7) in terms of simultaneous equations and making use of the linear operator, we have

$$P_{0,1} = \frac{P_{0,0}}{\rho} \tag{8}$$

$$P_{1,1} = \frac{P_{0,0}}{2} \tag{9}$$

$$P_{b,1} = \frac{P_{0,0}}{2} \tag{10}$$

$$P_{1,0} = \frac{P_{0,0}}{2\rho} \tag{11}$$

$$P_{b,1} = \left[\frac{1+\rho}{\rho} - \frac{1}{2\rho}\right] P_{0,0} \tag{12}$$

In view of (10) and (12), it is fairly easy to observe that $\rho < 1$.
Again, in light of necessary condition (3), we readily arrive at

$$P_{0,1} = -2, P_{0,0} = 2, P_{1,1} = 1, P_{b,1} = 1, P_{1,0} = -1$$

Now the expected number of patents in the system will be.

$$L_s = 0.P_{0,0} + 1.(P_{0,1} + P_{1,0}) + 2(P_{1,1} + P_{b,1})$$

$$= 0 + 1\left(\frac{P_{0,0}}{\rho} + \frac{P_{0,0}}{2\rho}\right) - 2\left(\frac{P_{0,0}}{2} + \frac{P_{0,0}}{2}\right)$$

$$= \frac{P_{0,0}}{\rho} + \frac{P_{0,0}}{2\rho} + 2P_{0,0}$$

$$= P_{0,0}\left(\frac{3+4\rho}{2\rho}\right)$$

$$= 4 + \frac{3}{\rho}, \quad \text{for} \quad P_{0,0} = 2$$

Again expected waiting time in system

$$Ws = \frac{Ls}{\lambda} = \frac{1}{\lambda}\left(4 + \frac{3}{\rho}\right)$$

And expected waiting time in queue

$$Wq = Ws - \frac{1}{\mu} = \frac{1}{\lambda}\left(4 + \frac{3}{\rho}\right) - \frac{1}{\mu}$$

While expected number in the queue

$$Lq = \lambda Wq = \lambda\left[\frac{1}{\lambda}\left(4 + \frac{3}{\rho}\right) - \frac{1}{\mu}\right]$$

$$= 4 + \frac{3}{\rho} - \frac{\lambda}{\mu}$$

$$= 4 + \frac{3-\rho^2}{\rho}$$

The steady state waiting time W_∞ has the distribution

$$P(W_\infty > t) = \rho e^{-(\mu-\lambda)t}, t > 0$$

So that

$$Var(W_\infty) = \frac{\rho(2-\rho)}{\mu^2(1-\rho)^2}$$

In contrast, assuming that I_∞ is the number of customers in the system,
So that

$$P(I_{\infty=K}) = (1-\rho)\rho^K, \quad K \geq 0$$

$$E\left[Var\left(\frac{W_\infty}{I_\infty}\right)\right] = \sum_{K=0}^{\infty}(1-\rho)\rho^K\frac{K}{\mu^2}$$

$$\frac{\rho}{\mu^2(1-\rho)} \quad \text{and} \quad \frac{E[Var(W_\infty/I_\infty)]}{Var(W_\infty)} = \frac{1-\rho}{2-\rho} \tag{13}$$

From (13) we see that the variance ratio is decreasing in , starting at 1/2 at $\rho = 0$ and is asymptotically $(1-\rho)$ as $\rho \to 1$. Hence conditioning provides a big variance reduction in heavy traffic. Also it is noteworthy that in those rare instances in which the number k of patients in the system is large, and delays will tend to be large, the prediction is reliable and differs dramatically from the steady-state mean. Thus the value of delay prediction actually may be much greater than predicted by (13).

The arrival process of patients to the M/G/I queue is Poisson with rate λ, and the service times are i.i.d. random variables with finite first and second moments. The service time distribution is denoted as F and the service rate is denoted as $\mu = 1/E[S]$.

Since, The expected length of a cycle is given by Ross [3] for example)

$$\frac{E[S]}{(1-\rho)} + \frac{1}{\lambda} = \frac{1}{\lambda(1-\rho)}$$

The M/G/1 system regenerates after the end of each cycle.

Lest S be a random variable that is independent of W and has the distribution F. The expected length of a busy period is given by $E[S]/(1-\rho)$, see Wolf [4]. An idle period followed by a busy period will be called a cycle.

References

[1] Donald Gross and Carl M. Harris (1974), Fundamentals of Queueing Theory, John Wiley and Sons, New York.

[2] Pandey ,O.N. (1998), Ph.D. Thesis, D.D.U. Gorakhpur University.

[3] Ross, S.M. (1983) Stochastic Process, 2nd Edn. Norht Holland.

[4] Wolf, R.W. (1989) Stochasti Modelling and the theory of Queues, Prentice Hall, Englewood Cliffs, NJ.