J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.5, No.2 (2016), pp. 75-88

STUDY OF CERTAIN MOCK THETA FUNCTIONS AND SOME PARTIAL ORDER RELATIONS

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Abstract: This paper consists representations of different order of mock theta functions and relations and expansions of partial mock theta functions, mock theta functions of tenth, third, fifth and sixth order. Using a simple identity, we connect the tenth order mock theta functions with partial tenth order mock theta functions of third, fifth, and sixth order. In this paper, we study relations between fifth order mock theta functions and third order mock theta functions and their partial sums. we have studied expansions of a tenth order mock theta functions in terms of partial mock theta function of tenth order also.
In this paper, we have established some new relations between mock theta functions and partial mock theta functions using tenth, third, fifth, and sixth order mock theta functions.

Keywords and Phrases: Basic Hypergeometric Series, Mock-Theta Functions, Partial Mock-Theta Functions, Identity

2010 Mathematics Subject Classification: 33D15, 11B65, 11A55.

1. Introduction

Ramanujan's definition of a mock theta function and notations The following q-notations have been used. For $|q| < 1$ and $|q^k| < 1$,

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad n \geq 1$$

$$(a;q^k)_n = \prod_{j=0}^{n-1}(1-aq^{kj}), \quad n \geq 1$$

$$(a;q)_0 = (a;q^k)_0 = 1$$
$$(a;q^k)_\infty = \prod_{j=0}^{\infty}(1-aq^{kj})$$
$$(a)_n = (a;q)_n$$

A generalized basic hypergeometric series with base q_1 is defined as

$${}_j\phi_k(a_1, a_2, ..., a_j; b_1, ..., b_k; q; z)$$

$${}_j\phi_k\left[\begin{matrix} a_1, a_2, ..., a_j \\ b_1, b_2, ..., b_k \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_j; q)_n}{(b_1, b_2, ..., b_k, q; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+k-j} z^n$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $j > k+1$.

In his famous "deathbed letter, Ramanujan introduced the notion of a mock theta function.

Following Andrews and Hickerson and Zwegers we give the following version of Ramanujan's definition.

A mock theta function is a function f of the complex variable q, defined by a q-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

(1) infinitely many roots of unity are exponential singularities,

(2) for every root of unity ξ there is a theta function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as q tends to ξ radially,

(3) f is not the sum of two functions, one of which is a theta function and the other a function which is bounded radially toward all roots of unity.

If $M(q) = \sum_{n=0}^{\infty} \Omega_n$ is a mock theta function, then the corresponding partial mock theta function is denoted by the terminating series,

$$M_r(q) = \sum_{n=0}^{r} \Omega_n$$

The complete list of mock theta functions of order 3 are

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2...(1+q^n)^2} \tag{1.1}$$

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)^2...(1+q^{2n})} \tag{1.2}$$

$$\Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^3)^2...(1+q^{2n})} \tag{1.3}$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)...(1-q^n+q^{2n})} \tag{1.4}$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n-1)}}{(1-q)^2(1-q^3)^2...(1-q^{2n+1})^2} \tag{1.5}$$

$$\upsilon(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1+q)(1+q^3)...(1-q^{2n+1})} \tag{1.6}$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^5)...(1+q^{2n+1}+q^{4n+2})} \tag{1.7}$$

Ramanujan gave 10 mock theta functions of order five which are given by

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n} \tag{1.8}$$

$$1+2\Psi_0(q) = \sum_{n=0}^{\infty} (-1;q)_n q^{\left(\begin{array}{l} n+1 \\ 2 \end{array}\right)} \tag{1.9}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_{\infty}} \tag{1.10}$$

$$\Phi_0(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{n^2} \tag{1.11}$$

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q)_n} \tag{1.12}$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}} \tag{1.13}$$

$$\Phi_1(q) = \sum_{n=0}^{\infty}(-q;q^2)_n q^{(n+1)^2} \tag{1.14}$$

$$\Psi_1(q) = \sum_{n=0}^{\infty}(-q)_n q^{\binom{n+1}{2}} \tag{1.15}$$

$$\chi_0(q) = 2F_0(q) - \Phi_0(-q) \tag{1.16}$$

$$\chi_1(q) = 2F_1(q) + q^{-1}\Phi_1(-q) \tag{1.17}$$

Ramanujan gave seven mock theta functions of order six, given by

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q)_{2n}} \tag{1.18}$$

$$\Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q)_{2n+1}} \tag{1.19}$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}(-q)_n}{(q;q^2)_{n+1}} \tag{1.20}$$

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+2}{2}}(-q)_n}{(q;q^2)_{n+1}} \tag{1.21}$$

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q)_n} \tag{1.22}$$

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q)_n} \tag{1.23}$$

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q)_n}{(q^3;q^3)_n} \tag{1.24}$$

(Andrews and Hickerson 1991).
Complete list of Tenth order mock theta functions are given by

$$\Phi_R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}} \tag{1.25}$$

$$\Psi_R(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+1)/2}}{(q;q^2)_{n+1}} \tag{1.26}$$

$$X_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \tag{1.27}$$

$$\chi_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} \tag{1.28}$$

2. Main Identity

We give a proof of the following identity

$$\sum_{j=0}^{p} \alpha_j \beta_j = \beta_{p+1} \sum_{j=0}^{p} \alpha_j + \sum_{m=0}^{p} (\beta_m - \beta_{m+1}) \sum_{j=0}^{m} \alpha_j \tag{2.1}$$

Proof: The above identity can be proved by simply rearrangement of series

$$\sum_{m=0}^{p} (\beta_m - \beta_{m+1}) \sum_{j=0}^{m} \alpha_j = (\beta_0 - \beta_1)\alpha_0 + (\beta_1 - \beta_2) \sum_{j=0}^{1} \alpha_j + (\beta_2 - \beta_3) \sum_{j=0}^{2} \alpha_j$$

$$+ ... + (\beta_p - \beta_{p+1}) \sum_{j=0}^{p} \alpha_j$$

$$\Rightarrow \alpha_0\beta_0 + \alpha_1\beta_1 + ... + \alpha_p\beta_p - \beta_{p+1}(\alpha_0 + \alpha_1 + ... + \alpha_p)$$

$$\Rightarrow \sum_{j=0}^{p} \alpha_j \beta_j - \beta_{p+1} \sum_{j=0}^{p} \alpha_j$$

which proves (2.1). Now, we'll derive and show some partial relations using different order of mock theta functions

3. Results

3(a). Results Using Tenth Order Mock Theta Functions

(i) Taking $\alpha_j = \dfrac{q^{(j+1)(j+2)/2}}{(q;q^2)_{j+1}}$, $\beta_j = (q;q^2)_{j+1}$ in (2.1), we get

$$\Phi_1(q) = (q;q^2)_{p+2} \Psi_{Rp}(q) + \sum_{m=0}^{p} q^{2m+3} (q;q^2)_{m+1} \Psi_{Rm}(q) \tag{3.1}$$

where $\Phi_1(q) = \sum_{j=0}^{p} q^{(j+1)(j+2)/2}$.
(ii) Letting $p \to \infty$ in (i), we get

$$\Phi_1(q) = (q;q^2)_\infty \Psi_R(q) + \sum_{m=0}^{\infty} q^{2m+3}(q;q^2)_{m+1}\Psi_{Rm}(q) \tag{3.2}$$

(iii) Taking $\alpha_j = \dfrac{(-1)^j q^{j^2}}{(-q;q)_{2j+1}}$, $\beta_j = \dfrac{q^{2j+1}}{(1+q^{2j+1})}$ in (2.1) we get

$$\chi_{Rp}(q) = \frac{q^{2p+3}}{(1+q^{2p+3})} X_{Rp}(q) + \sum_{m=0}^{p} \left[\frac{q^{2m+1}(1-q^2)}{(1+q^{2m+1})}\right] X_{Rm}(q) \tag{3.3}$$

(iv) Letting $p \to \infty$ in (iii), we get

$$\chi_R(q) = \sum_{m=0}^{\infty} \left[\frac{q^{2m+1}(1-q^2)}{(1+q^{2m+1})}\right] X_{Rm}(q) \tag{3.4}$$

(v) Taking $\alpha_j = \dfrac{(-1)^j q^{(j+1)^2}}{(-q;q)_{2j+1}}$, $\beta_j = \dfrac{1+q^{2j+1}}{q^{2j+1}}$ in (2.1) we get

$$X_{Rp}(q) = \frac{1+q^{2p+3}}{q^{2p+3}} \chi_{Rp}(q) + \sum_{m=0}^{p} \left[\frac{(q^2-1)}{q^{2m+3}}\right] \chi_{Rm}(q) \tag{3.5}$$

(vi) Letting $p \to \infty$ in (v), we get

$$X_R(q) = \sum_{m=0}^{\infty} \left[\frac{(q^2-1)}{q^{2m+3}}\right] \chi_{Rm}(q) \tag{3.6}$$

3(b). Results Using Tenth and Sixth Order Mock Theta Functions

(i) Taking $\alpha_j = \dfrac{(-1)^j q^{j^2}(q;q^2)_j}{(-q;q)_{2j}}$, $\beta_j = \dfrac{q^{2j+1}}{(1+q^{2j+1})(q;q^2)_j}$ in (2.1), we get

$$\chi_{Rp}(q) = \frac{q^{2p+3}}{(1+q^{2p+3})(q;q^2)_{p+1}} \Phi_{Lp}(q)$$

$$+ \sum_{m=0}^{p} \frac{q^{2m+1}}{(q;q^2)_{m+1}} \left(\frac{1-q^{2m+1}-q^{4m+4}-q^2}{(1+q^{2m+1})(1+q^{2m+3})}\right) \Phi_{Lm}(q) \tag{3.7}$$

(ii) Letting $p \to \infty$ in (i), we get

$$\chi_R(q) = \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q;q^2)_{m+1}} \left(\frac{1 - q^{2m+1} - q^{4m+4} - q^2}{(1+q^{2m+1})(1+q^{2m+3})} \right) \Phi_{Lm}(q) \tag{3.8}$$

(iii) Taking $\alpha_j = \dfrac{(-1)^j q^{(j+1)^2}(q;q^2)_j}{(-q;q)_{2j+1}}$, $\beta_j = \dfrac{(1+q^{2j+1})}{q^{2j+1}(q;q^2)_j}$ in (2.1), we get

$$\chi_{Lp}(q) = \frac{(1+q^{2p+3})}{(q^{2p+3})(q;q^2)_{p+1}} \Psi_{Lp}(q)$$
$$+ \sum_{m=0}^{p} \frac{1}{q^{2m+1}(q;q^2)_{m+1}} \left[\frac{q^2 - q^{4m+4} - 1 - q^{2m+3}}{q^2} \right] \Psi_{Lm}(q) \tag{3.9}$$

(iv) Letting $p \to \infty$ in (iii), we get

$$X_L(q) = \sum_{m=0}^{\infty} \frac{1}{q^{2m+1}(q;q^2)_{m+1}} \left[\frac{q^2 - q^{4m+4} - 1 - q^{2m+3}}{q^2} \right] \Psi_{Lm}(q) \tag{3.10}$$

(v) Taking $\alpha_j = \dfrac{q^{j(j+1)/2}(-q;q)_j}{(q;q^2)_{j+1}}$, $\beta_j = \dfrac{q^{j+1}}{(-q;q)_j}$ in (2.1), we get

$$\Psi_{Rp}(q) = \frac{q^{p+2}}{(-q;q)_{p+1}} \rho_{Lp}(q) + \sum_{m=0}^{p} \frac{q^{m+1}}{(-q;q)_{m+1}} [1 + q^{m+1} - q] \rho_{Lm}(q) \tag{3.11}$$

(vi) Letting $p \to \infty$ in (v), we get

$$\Psi_R(q) = \sum_{m=0}^{\infty} \frac{q^{m+1}}{(-q;q)_{m+1}} [1 + q^{m+1} - q] \rho_{Lm}(q) \tag{3.12}$$

(vii) Taking $\alpha_j = \dfrac{q^{j(j+1)(j+2)/2}(-q;q)_j}{(q;q^2)_{j+1}}$, $\beta_j = \dfrac{1}{(-q;q)_{j+1}}$ in (2.1), we get

$$\Phi_{Rp}(q) = \frac{1}{q^{p+2}(-q;q)_{p+1}} \sigma_{Lp}(q) + \sum_{m=0}^{p} \frac{1}{q^{m+1}(-q;q)_{m+1}} \left[1 + q^{m+1} - \frac{1}{q} \right] \sigma_{Lm}(q) \tag{3.13}$$

(viii) Letting $p \to \infty$ in (vii), we get

$$\Phi_R(q) = \sum_{m=0}^{\infty} \frac{1}{q^{m+1}(-q;q)_{m+1}} \left[1 + q^{m+1} - \frac{1}{q} \right] \sigma_{Lm}(q) \tag{3.14}$$

3(c). Results Using Fifth and Third Order Mock Theta Functions

(i) Taking $\alpha_j = \dfrac{q^{2j^2}}{(q;q^2)_j}$, $\beta_j = \dfrac{1}{q^{j^2}}$ in (2.1), we get

$$\Psi_P(q) = \frac{1}{q^{(p+1)^2}}\Psi_p(q) + \sum_{m=0}^{p} \frac{(q^{2m+1}-1)}{q^{(m+1)^2}}\Psi_m(q) \tag{3.15}$$

(ii) Letting $p \to \infty$ in (i), we get

$$\Psi(q) = \sum_{m=0}^{\infty} \frac{(q^{2m+1}-1)}{q^{(m+1)^2}}\Psi_m(q) \tag{3.16}$$

4. Proof of the above results

Proof of (3.1 and 3.2)

Taking $\alpha_j = \dfrac{q^{(j+1)(j+2)/2}}{(q;q^2)_{j+1}}$, $\beta_j = (q;q^2)_{j+1}$ in (2.1), we get

$$\sum_{j=0}^{p} \frac{q^{(j+1)(j+2)/2}}{(q;q^2)_{j+1}}(q;q^2)_{j+1} = (q;q^2)_{p+2}\sum_{j=0}^{p}\frac{q^{(j+1)(j+2)/2}}{(q;q^2)_{j+1}}$$

$$+\sum_{m=0}^{p}\left((q;q^2)_{m+1} - (q;q^2)_{m+2}\right)\sum_{j=0}^{m}\frac{q^{(j+1)(j+2)/2}}{(q;q^2)_{j+1}}$$

$$\Rightarrow \sum_{j=0}^{p} q^{(j+1)(j+2)/2} = (q;q^2)_{p+2}\Psi_{Rp}(q) + \sum_{m=0}^{p}(q;q^2)_{m+1}(1-1+q^{2m+2+1})\Psi_{Rm}(q)$$

$$\Rightarrow \Phi_1(q) = (q;q^2)_{p+2}\Psi_{Rp}(q) + \sum_{m=0}^{p} q^{2m+3}(q;q^2)_{m+1}\Psi_{Rm}(q)$$

where $\phi_1(q) = \sum_{j=0}^{p} q^{(j+1)(j+2)/2}$ and letting $p \to \infty$ we get

$$\Phi_1(q) = (q;q^2)_{p+2}\Psi_R(q) + \sum_{m=0}^{\infty} q^{2m+3}(q;q^2)_{m+1}\Psi_{Rm}(q).$$

Proof of (3.3 and 3.4)

Taking $\alpha_j = \dfrac{(-1)^j q^{j^2}}{(-q;q)_{2j+1}}$, $\beta_j = \dfrac{q^{2j+1}}{(1+q^{2j+1})}$ in (2.1), we get

$$\sum_{j=0}^{p} \frac{(-1)^j q^{j^2}}{(-q;q)_{2j+1}} \frac{q^{2j+1}}{(1+q^{2j+1})} = \frac{q^{2p+3}}{(1+q^{2p+3})} \sum_{j=0}^{p} \frac{(-1)^j q^{j^2}}{(-q;q)_{2j+1}}$$

$$+ \sum_{m=0}^{p} \left(\frac{q^{2m+1}}{(1+q^{2m+1})} - \frac{q^{2m+3}}{(1+q^{2m+3})} \right) \sum_{j=0}^{m} \frac{(-1)^j q^{j^2}}{(-q;q)_{2j+1}}$$

$$\Rightarrow \sum_{j=0}^{p} \frac{(-1)^j q^{(j+1)^2}}{(-q;q)_{2j+1}} = \frac{q^{2p+3}}{(1+q^{2p+3})} X_{Rp}(q)$$

$$+ \sum_{m=0}^{p} \left[\frac{q^{2m+1} + q^{4m+4} - q^{2m+3} - q^{4m+6}}{(1+q^{2m+1})(1+q^{2m+3})} \right] X_{Rm}(q)$$

$$\Rightarrow \chi_{Rp}(q) = \frac{q^{2p+3}}{(1+q^{2p+3})} X_{Rp}(q) + \sum_{m=0}^{p} \left[\frac{(1-q^2)(q^{2m+1} + q^{4m+4})}{(1+q^{2m+1})(1+q^{2m+3})} \right] X_{Rm}(q)$$

and letting $p \to \infty$ in above we get

$$\chi_R(p) = \sum_{m=0}^{\infty} \left[\frac{q^{2m+1}(1-q^2)}{(1+q^{2m+1})} \right] X_{Rm}(q)$$

Proof of (3.5 and 3.6)

Taking $\alpha_j = \dfrac{(-1)^j q^{(j+1)^2}}{(-q;q)_{2j+1}}$, $\beta_j = \dfrac{(1+q^{2j+1})}{q^{2j+1}}$ in (2.1), we get

$$\sum_{j=0}^{p} \frac{(-1)^j q^{(j+1)^2}}{(-q;q)_{2j+1}} \frac{(1+q^{2j+1})}{q^{2j+1}} = \frac{(1+q^{2p+3})}{q^{2p+3}} \sum_{j=0}^{p} \frac{(-1)^j q^{(j+1)^2}}{(-q;q)_{2j+1}}$$

$$+ \sum_{m=0}^{p} \left(\frac{(1+q^{2m+1})}{q^{2m+1}} - \frac{(1+q^{2m+3})}{q^{2m+3}} \right) \sum_{j=0}^{m} \frac{(-1)^j q^{(j+1)^2}}{(-q;q)_{2j+1}}$$

$$\Rightarrow \sum_{j=0}^{p} \frac{(-1)^j q^{j^2}}{(-q;q)_{2j}} = \frac{(1+q^{2p+3})}{q^{2p+3}} \chi_{Rp}(q)$$

$$+ \sum_{m=0}^{p} \left[\frac{q^{2m+3} + q^{4m+4} - q^{2m+1} - q^{4m+4}}{q^{2m+1} q^{2m+3}} \right] \chi_{Rm}(q)$$

$$\Rightarrow X_{Rp}(q) = \frac{(1+q^{2p+3})}{q^{2p+3}}\chi_{Rp}(q) + \sum_{m=0}^{p}\left[\frac{q^{2m+3}-q^{2m+1}}{q^{2m+1}q^{2m+3}}\right]\chi_{Rm}(q)$$

$$\Rightarrow X_{Rp}(q) = \frac{(1+q^{2p+3})}{q^{2p+3}}\chi_{Rp}(q) + \sum_{m=0}^{p}\left[\frac{(q^2-1)}{q^{2m+3}}\right]\chi_{Rm}(q)$$

and letting $p \to \infty$ in (xii), we get

$$X_R(q) = \sum_{m=0}^{\infty}\left[\frac{(q^2-1)}{q^{2m+3}}\right]X_{Rm}(q).$$

Proof of (3.7) and (3.8)

Taking $\alpha_j = \dfrac{(-1)^j q^{j^2}(q;q^2)_j}{(-q;q)_{2j}}$, $\beta_j = \dfrac{q^{2j+1}}{(1+q^{2j+1})(q;q^2)_j}$ in (2.1), we get

$$\sum_{j=0}^{p}\frac{(-1)^j q^{j^2}(q;q^2)_j}{(-q;q)_{2j}}\frac{q^{2j+1}}{(1+q^{2j+1})(q;q^2)_j} = \frac{q^{2p+3}}{(1+q^{2p+3})(q;q^2)_{p+1}}\sum_{j=0}^{p}\frac{(-1)^j q^{j^2}(q;q^2)_j}{(-q;q)_{2j}}$$

$$+\sum_{m=0}^{p}\left(\frac{q^{2m+1}}{(1+q^{2m+1})(q;q^2)_m} - \frac{q^{2m+3}}{(1+q^{2m+3})(q;q^2)_{m+1}}\right)\sum_{j=0}^{m}\frac{(-1)^j q^{j^2}(q;q^2)_j}{(-q;q)_{2j}}$$

$$\Rightarrow \sum_{j=0}^{p}\frac{(-1)^j q^{(j+1)^2}}{(-q;q)_{2j+1}} = \frac{q^{2p+3}}{(1+q^{2p+3})(q;q^2)_{p+1}}\Phi_{Lp}(q)$$

$$+\sum_{m=0}^{p}\frac{q^{2m+1}}{(q;q^2)_{m+1}}\left[\frac{(1-q^{2m+1})}{(1+q^{2m+1})} - \frac{q^2}{(1+q^{2m+3})}\right]\Phi_{Lm}(q)$$

$$\Rightarrow \chi_{Rp}(q) = \frac{q^{2p+3}}{(1+q^{2p+3})(q;q^2)_{p+1}}\Phi_{Lp}(q)$$

$$+\sum_{m=0}^{p}\frac{q^{2m+1}}{(q;q^2)_{m+1}}\left[\frac{(1-q^{2m+1})(1+q^{2m+3}) - q^2(1+q^{2m+1})}{(1+q^{2m+1})(1+q^{2m+3})}\right]\Phi_{Lm}(q)$$

$$\Rightarrow \chi_{Rp}(q) = \frac{q^{2p+3}}{(1+q^{2p+3})(q;q^2)_{p+1}}\Phi_{Lp}(q)$$

$$+\sum_{m=0}^{p}\frac{q^{2m+1}}{(q;q^2)_{m+1}}\left[\frac{1-q^{2m+1}-q^{4m+4}-q^2}{(1+q^{2m+1})(1+q^{2m+3})}\right]\Phi_{Lm}(q)$$

and letting $p \to \infty$, we get

$$\chi_R(q) = \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q;q^2)_{m+1}} \left[\frac{1 - q^{2m+1} - q^{4m+4} - q^2}{(1+q^{2m+1})(1+q^{2m+3})} \right] \Phi_{Lm}(q)$$

Proof of (3.9) and (3.10)

Taking $\alpha_j = \dfrac{(-1)^j q^{(j+1)^2}(q;q^2)_j}{(-q;q)_{2j+1}}$, $\beta_j = \dfrac{(1+q^{2j+1})}{q^{2j+1}(q;q^2)_j}$ in (2.1), we get

$$\sum_{j=0}^{p} \frac{(-1)^j q^{(j+1)^2}(q;q^2)_j}{(-q;q)_{2j+1}} \frac{(1+q^{2j+1})}{q^{2j+1}(q;q^2)_j} = \frac{(1+q^{2p+3})}{q^{2p+3}(q;q^2)_{p+1}} \sum_{j=0}^{p} \frac{(-1)^j q^{(j+1)^2}(q;q^2)_j}{(-q;q)_{2j+1}}$$

$$+ \sum_{m=0}^{p} \left(\frac{(1+q^{2m+1})}{q^{2m+1}(q;q^2)_m} - \frac{(1+q^{2m+3})}{q^{2m+3}(q;q^2)_{m+1}} \right) \sum_{j=0}^{m} \frac{(-1)^j q^{(j+1)^2}(q;q^2)_j}{(-q;q)_{2j+1}}$$

$$\Rightarrow \sum_{j=0}^{p} \frac{(-1)^j q^{j^2}}{(-q;q)_{2j}} = \frac{(1+q^{2p+3})}{q^{2p+3}(q;q^2)_{j+1}} \Psi_{Lp}(q)$$

$$+ \sum_{m=0}^{p} \frac{1}{q^{2m+1}(q;q^2)_{m+1}} \left[(1+q^{2m+1})(1-q^{2m+1}) - \frac{(1+q^{2m+3})}{q^2} \right] \Psi_{Lm}(q)$$

$$\Rightarrow X_{Lp}(q) = \frac{(1+q^{2p+3})}{q^{2p+3}(q;q^2)_{p+1}} \Psi_{Lp}(q)$$

$$+ \sum_{m=0}^{p} \frac{1}{q^{2m+1}(q;q^2)_{m+1}} \left[\frac{q^2 - q^{4m+4} - 1 - q^{2m+3}}{q^2} \right] \Psi_{Lm}(q)$$

and letting $p \to \infty$, we get

$$X_L(q) = \sum_{m=0}^{\infty} \frac{1}{q^{2m+1}(q;q^2)_{m+1}} \left[\frac{q^2 - q^{4m+4} - 1 - q^{2m+3}}{q^2} \right] \Psi_{Lm}(q)$$

Proof of 3.11 and 3.12 Taking $\alpha_j = \dfrac{(-1)^j q^{j(j+1)/2}(-q;q)_j}{(q;q^2)_{j+1}}$, $\beta_j = \dfrac{q^{j+1}}{(-q;q)_j}$ in (2.1), we get

$$\sum_{j=0}^{p} \frac{q^{j(j+1)/2}(-q;q)_j}{(q;q^2)_{j+1}} \frac{q^{j+1}}{(-q;q)_j} = \frac{q^{p+2}}{(-q;q)_{p+1}} \sum_{j=0}^{p} \frac{q^{j(j+1)/2}(-q;q)_j}{(q;q^2)_{j+1}}$$

$$+\sum_{m=0}^{p}\left[\frac{q^{m+1}}{(-q;q)_m}-\frac{q^{m+2}}{(-q;q)_{m+1}}\right]\sum_{j=0}^{m}\frac{q^{j(j+1)/2}(-q;q)_j}{(q;q^2)_{j+1}}$$

$$\Rightarrow\sum_{j=0}^{p}\frac{q^{(j+1)(j+2)/2}}{(q;q^2)_{j+1}}=\frac{q^{p+2}}{(-q;q)_{p+1}}\rho_{Lp}(q)+\sum_{m=0}^{p}\frac{q^{m+1}}{(-q;q)_{m+1}}[1+q^{m+1}-q]\rho_{Lm}(q)$$

$$\Rightarrow\Psi_{Rp}(q)=\frac{q^{p+2}}{(-q;q)_{p+1}}\rho_{Lp}(q)+\sum_{m=0}^{p}\frac{q^{m+1}}{(-q;q)_{m+1}}[1+q^{m+1}-q]\rho_{Lm}(q)$$

and letting $p\to\infty$, we get

$$\Psi_R(q)=\sum_{m=0}^{\infty}\frac{q^{m+1}}{(-q;q)_{m+1}}[1+q^{m+1}-q]\rho_{Lm}(q)$$

Proof of 3.13 and 3.14 Taking $\alpha_j=\dfrac{q^{(j+1)(j+2)/2}(-q;q)_j}{(q;q^2)_{j+1}}$, $\beta_j=\dfrac{1}{q^{j+1}(-q;q)_j}$ in (2.1), we get

$$\sum_{j=0}^{p}\frac{q^{(j+1)(j+2)/2}(-q;q)_j}{(q;q^2)_{j+1}}\frac{1}{q^{j+1}(-q;q)_j}=\frac{1}{q^{p+2}(-q;q)_{p+1}}\sum_{j=0}^{p}\frac{q^{(j+1)(j+2)/2}(-q;q)_j}{(q;q^2)_{j+1}}$$

$$+\sum_{m=0}^{p}\left[\frac{1}{q^{m+1}(-q;q)_m}-\frac{1}{q^{m+2}(-q;q)_{m+1}}\right]\sum_{j=0}^{m}\frac{q^{(j+1)(j+2)/2}(-q;q)_j}{(q;q^2)_{j+1}}$$

$$\Rightarrow\sum_{j=0}^{p}\frac{q^{j(j+1)/2}}{(q;q^2)_{j+1}}=\frac{1}{q^{p+2}(-q;q)_{p+1}}\sigma_{Lp}(q)+\sum_{m=0}^{p}\frac{1}{q^{m+1}(-q;q)_m}[1+q^{m+1}-\frac{1}{q}]\sigma_{Lm}(q)$$

$$\Rightarrow\Phi_{Rp}(q)=\frac{1}{q^{p+2}(-q;q)_{p+1}}\sigma_{Lp}(q)+\sum_{m=0}^{p}\frac{1}{q^{m+1}(-q;q)_m}\left[\frac{q+q^{m+2}-1}{q}\right]\sigma_{Lm}(q)$$

and letting $p\to\infty$, we get

$$\Phi_R(q)=\sum_{m=0}^{p}\frac{1}{q^{m+1}(-q;q)_m}\left[\frac{q+q^{m+2}-1}{q}\right]\sigma_{Lm}(q)$$

Proof of (3.15) and (3.16)

Taking $\alpha_j=\dfrac{q^{2j^2}}{(q;q^2)_j}$, $\beta_j=\dfrac{1}{q^{j^2}}$ in (2.1), we get

$$\sum_{j=0}^{p}\frac{q^{2j^2}}{(q;q^2)_j}\frac{1}{q^{j^2}}=\frac{1}{q^{(p+1)^2}}\sum_{j=0}^{p}\frac{q^{2j^2}}{(q;q^2)_j}+\sum_{m=0}^{p}\left[\frac{1}{q^{m^2}}-\frac{1}{q^{(m+1)^2}}\right]\sum_{j=0}^{m}\frac{q^{2j^2}}{(q;q^2)_j}$$

$$\Rightarrow \sum_{j=0}^{p} \frac{q^{j^2}}{(q;q^2)_j} = \frac{1}{q^{(p+1)^2}} \Psi_p(q) + \sum_{m=0}^{p} \left(\frac{q^{(m+1)^2} - q^{m^2}}{q^{m^2} q^{(m+1)^2}} \right) \Psi_m(q)$$

$$\Rightarrow \Psi_p(q) = \frac{1}{q^{(p+1)^2}} \Psi_p(q) + \sum_{m=0}^{p} \frac{q^{(2m+1)} - 1}{q^{(m+1)^2}} \Psi_m(q)$$

Letting $p \to \infty$, we get

$$\Psi(q) = \sum_{m=0}^{\infty} \frac{(q^{2m+1} - 1)}{q^{(m+1)^2}} \Psi_m(q)$$

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