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ON CERTAIN DOUBLE SERIES IDENTITIES

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Abstract: In this paper, making use of Bailey transform and WP-Bailey transform, certain double series identities of Rogers-Ramanujan type have been established.

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1. Introduction, Notations and Definitions

In the present paper, we adopt the following notations and definitions. The q-rising factorial is defined by, for $|q| < 1$.

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n = 1, 2, 3, ...$$

$$(a;q)_0 = 1$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r)$$

and

$$(a_1, a_2, ..., a_r; q)_n = (a_1;q)_n(a_2;q)_n...(a_r;q)_n.$$

With theses notations, a basic hypergeometric series (q-series) is defined by,

$${}_r\Phi_s\left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}.$$

Bailey [3,4] introduced a lemma which is simple but very useful. It states; If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.1}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \tag{1.2}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.3}$$

where $\alpha_r, \delta_r, u_r, v_r$ are arbitrary sequences of r alone.
If we put the value of β_n from (1.1) in (1.3) and apply the identity,

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} A(n,r) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} A(n+r,r). \tag{1.4}$$

[5; Lemma (2.1 (2)) p. 100]

Then new form of the equation (1.3) becomes

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \alpha_r u_n \delta_{r+n} v_{n+2r}. \tag{1.5}$$

In order to make application of the lemma, Bailey choose $u_r = \frac{1}{(q;q)_r}$, $v_r = \frac{1}{(aq;q)_r}$ which gives the following result.

Theorem 1

If

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{1.6}$$

$$\gamma_n = \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{(q;q)_r(aq;q)_{r+2n}}, \tag{1.7}$$

Then under suitable convergence conditions,

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{\alpha_r\delta_{n+r}}{(q;q)_n(aq;q)_{n+2r}}. \tag{1.8}$$

Taking $\delta_r=(\rho_1,\rho_2;q)_r\left(\frac{aq}{\rho_1\rho_2}\right)^r$ in (1.7) we get

$$\gamma_n=\frac{(\rho_1,\rho_2;q)_n}{(aq;q)_{2n}}\left(\frac{aq}{\rho_1\rho_2}\right)^n {}_2\Phi_1\left[\begin{matrix}\rho_1q^n,\rho_2q^n;q;\frac{aq}{\rho_1\rho_2}\\ aq^{1+2n}\end{matrix}\right]. \tag{1.9}$$

Summing the ${}_2\Phi_1$ series by using [6; App. IV (IV.2)] we have

$$\gamma_n=\frac{(aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2;q)_\infty}\frac{(\rho_1,\rho_2;q)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1,aq/\rho_2;q)_n}. \tag{1.10}$$

Putting these values of γ_n and δ_n in (1.8) we finally get,

$$\frac{(aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n\alpha_n$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\rho_1,\rho_2;q)_{r+n}(aq/\rho_1\rho_2)^{r+n}}{(q;q)_n(aq;q)_{n+2r}}\alpha_r, \tag{1.11}$$

provided the infinite series are convergent and $\alpha_0=1$.
Taking $\rho_1,\rho_2\to\infty$, (1.11) yileds

$$\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}a^nq^{n^2}\alpha_n=\sum_{n,r=0}^{\infty}\frac{q^{(n+r)^2}a^{n+r}\alpha_r}{(q;q)_n(aq;q)_{n+2r}}. \tag{1.12}$$

2. Special cases of (1.12)

In this section we shall establish certain double series identities by making use of (1.12)

(i) Taking $\alpha_n=\frac{1}{(q;q)_n}$ in (1.12) we find,

$$(aq;q)_\infty\sum_{n,r=0}^{\infty}\frac{q^{(n+r)^2}a^{n+r}}{(q;q)_n(q;q)_r(aq;q)_{n+2r}}=\sum_{n=0}^{\infty}\frac{q^{n^2}a^n}{(q;q)_n}. \tag{2.1}$$

(ii) Choosing $a = 1$ in (2.1) and using [1; (10.1.1) p. 241] we get

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2}}{(q;q)_n (q;q)_r (q;q)_{n+2r}} = \frac{1}{(q,q^4;q^5)_\infty}. \tag{2.2}$$

(iii) Taking $a = q$ in (2.1) and using [1; (10.1.2) p. 241] we have,

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)(n+r+1)}}{(q;q)_n (q;q)_r (q;q)_{n+2r+1}} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{2.3}$$

(iv) Choosing $a = 1, \alpha_r = \dfrac{1}{q^r (q;q)_r}$ in (1.12) we obtain,

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2-r}}{(q;q)_n (q;q)_r (q;q)_{n+2r}} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(q;q)_n}$$

$$= \frac{1}{(q,q^4;q^5)_\infty} + \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{2.4}$$

(v) Taking $a = q$ and $\alpha_r = 1$ in (1.12) and using [1; (1.1.7) p. 11] we get,

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)(n+r+1)}}{(q;q)_n (q;q)_{n+2r+1}} = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}. \tag{2.5}$$

(vi) Taking $a = 1, \alpha_r = \dfrac{1}{(q;q)_{2r}}$ in (1.12) and using [1; (11.2.1) p. 252] we obtain,

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2}}{(q;q)_n (q;q)_{2r} (q;q)_{n+2r}} = \frac{1}{(q;q^2)_\infty (q^4,q^{16};q^{20})_\infty}. \tag{2.6}$$

(vii) For $a = q, \alpha_r = \dfrac{1}{(q;q)_{2r+1}}$ and by an appeal of the identity [1; (11.2.2) p. 252], (1.12) yields;

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)(n+r+1)}}{(q;q)_n (q;q)_{n+2r+1} (q;q)_{2r+1}} = \frac{1}{(q;q^2)_\infty (q^2,-q^3;-q^5)_\infty}. \tag{2.7}$$

(viii) Taking $\alpha_r = \dfrac{1}{(q;q)_{2r}}, a = 1$ and using [1; (11.2.3) p. 252] we obtain,

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2}}{(q;q)_n (q;q)_{n+2r+1} (q;q)_{2r}} = \frac{1}{(q;q^2)_\infty (-q,q^4;-q^5)_\infty}. \tag{2.8}$$

(ix) Taking $\alpha_r = \dfrac{1}{(q;q)_{2r+1}}, a = q^2$ in (1.12) and making use of [1; (11.2.4) p. 252] we have,

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)(n+r+2)}}{(q;q)_n (q;q)_{2r+1} (q;q)_{n+2r+2}} = \frac{1}{(q;q^2)_\infty (q^8, q^{12}; q^{20})_\infty}. \tag{2.9}$$

(x) Taking $a = 1, \alpha_r = \dfrac{(-q;q^2)_r}{(q;q)_{2r}}$ in (1.12) and using [1; (11.3.4) p. 254] we obtain,

$$(q;q)_\infty^2 \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2}(-q;q^2)_r}{(q;q)_n (q;q)_{n+2r} (q;q)_{2r}} = (q^6;q^{12})_\infty (q^{12};q^{12})_\infty. \tag{2.10}$$

(xi) Taking $\alpha_r = \dfrac{1}{(q;q)_r (aq;q)_r}$ in (1.12) and using [1; (6.2.31), p. 152] we get,

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2} a^{n+r}}{(q;q)_n (q;q)_r (aq;q)_r (aq;q)_{n+2r}} = \frac{1}{(aq;q)_\infty^2}. \tag{2.11}$$

For $a = 1$, (2.11) yields

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2}}{(q;q)_n (q;q)_r^2 (q;q)_{n+2r}} = \frac{1}{(q;q)_\infty^2}. \tag{2.12}$$

For $a = -1$, (2.11) yields

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2}(-1)^{n+r}}{(q;q)_n (q^2;q^2)_r (-q;q)_{n+2r}} = \frac{1}{(-q;q)_\infty^2}. \tag{2.13}$$

(xii) Taking $\alpha_r = (-1)^r q^r$ in (1.12) and using [1; (6.2.29), p. 152] we get,

$$(aq;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2} a^{n+r} (-q)^r}{(q;q)_n (aq;q)_{n+2r}}$$

$$= \frac{1}{1+} \frac{aq^2}{1+} \frac{a(q^4 - q^2)}{1+} \frac{aq^6}{1+} \frac{a(q^8 - q^4)}{1+} \frac{aq^{10}}{1 + \ldots}. \tag{2.14}$$

For $a = -1$, (2.14) yields

$$(-q;q)_\infty \sum_{n,r=0}^{\infty} \frac{(-1)^n q^{(n+r)^2+r}}{(q;q)_n (-q;q)_{n+2r}}$$

$$= \frac{1}{1-}\frac{q^2}{1-}\frac{(q^4-q^2)}{1-}\frac{q^6}{1-}\frac{(q^8-q^4)}{1-}\frac{q^{10}}{1-...}. \tag{2.15}$$

Similarly, a large number of double series identities can be scored.

3. WP Bailey pair and double series identities

Andrews [1] changed the Bailey lemma by taking $u_r = \frac{(k/a;q)_r}{(q;q)_r}$ and $v_r = \frac{(k;q)_r}{(aq;q)_r}$ in the following form;

Theorem 2 If

$$\beta_n(a,k) = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}}\alpha_r(a,k) \tag{3.1}$$

and

$$\gamma_n(a,k) = \sum_{r=0}^{\infty} \frac{(k/a;q)_r(k;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}}\delta_{r+n}(a,k), \tag{3.2}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty}\alpha_n(a,k)\gamma_n(a,k) = \sum_{n=0}^{\infty}\delta_n(a,k)\beta_n(a,k). \tag{3.3}$$

Sequence $\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ satisfying (3.1) are called WP-Bailey pair and $\langle\gamma_n(a,k)$ $,\delta_n(a,k)\rangle$ satisfying (3.2) are called conjugate WP-Bailey pair.

If we put the value of β_n from (3.1) in (3.3) and apply the identity (1.4) we get,

$$\sum_{n=0}^{\infty}\alpha_n(a,k)\gamma_n(a,k) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}\delta_{n+r}(a,k)\alpha_r(a,k) \tag{3.4}$$

Choosing $\delta_r(a,k) = (a^2q/k^2)^r$ in (3.2) and summing the ${}_2\Phi_1$ series by using [6; App. IV (IV.2)] we get

$$\gamma_n(a,k) = \frac{(aq/k,a^2q/k;q)_\infty(k;q)_{2n}(a^2q/k^2)^n}{(aq,a^2q/k^2;q)_\infty(a^2q/k;q)_{2n}}. \tag{3.5}$$

Putting these values of $\gamma_n(a,k)$ and $\delta_n(a,k)$ in (3.4) we get,

$$\frac{(aq/k,a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty}\sum_{n=0}^{\infty}\frac{(k;q)_{2n}(a^2q/k^2)^n}{(a^2q/k;q)_{2n}}\alpha_n(a,k)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}\left(\frac{a^2q}{k^2}\right)^{n+r}\alpha_r(a,k), \tag{3.6}$$

provided all the infinite series converges and $\alpha_0(a,k)=1$.

As $k\to\infty$, (3.6) yields

$$(aq;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n^2+2r^2+2nr} a^{n+2r}\alpha_r(a,k)}{(q;q)_n(aq;q)_{n+2r}} = \sum_{n=0}^{\infty} q^{2n^2}a^{2n}\alpha_n(a,k). \tag{3.7}$$

4. Special Cases of (3.6) and (3.7)

(i) Taking $a=1, \alpha_n(a,k)=\dfrac{1}{(q^2;q^2)_n}$ in (3.7) and using [1; (10.1.1) p. 241] we get

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n^2+2r^2+2nr}}{(q;q)_n(q;q)_{n+2r}(q^2;q^2)_r} = \frac{1}{(q^2,q^8;q^{10})_\infty}. \tag{4.1}$$

(ii) Taking $a=q, \alpha_n(a,k)=\dfrac{1}{(q^2;q^2)_n}$ in (3.7) and using [1; (10.1.2) p. 241] we find

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n^2+2r^2+2nr+n+2r}}{(q;q)_n(q;q)_{n+2r+1}} = \frac{1}{(q^4,q^6;q^{10})_\infty}. \tag{4.2}$$

(iii) Taking $a=q^2, \alpha_r(a,k)=\dfrac{(-q^2;q^4)_r}{(q^4;q^4)_r}$ in (3.7) and using [7; (34)] we get

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n^2+2r^2+2nr+2n+4r}(-q^2;q^4)_r}{(q;q)_n(q;q)_{n+2r+2}(q^4;q^4)_r} = \frac{1}{(q^6,q^8,q^{10};q^{16})_\infty}. \tag{4.3}$$

(iv) Taking $a=1, \alpha_r(a,k)=\dfrac{(-q^2;q^4)_r}{(q^4;q^4)_r}$ and using the identity [7; (36)] we have

$$(q;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n^2+2r^2+2nr}(-q^2;q^4)_r}{(q;q)_n(q;q)_{n+2r}(q^4;q^4)_r} = \frac{1}{(q^2,q^8,q^{14};q^{16})_\infty}. \tag{4.4}$$

(v) Taking $\alpha_r(a,k)=\dfrac{(\alpha,\beta;q^2)_r}{(\gamma,q^2;q^2)_r}$ in (3.6) we find,

$$\frac{(aq/k,a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty}\,{}_4\Phi_3\left[\begin{matrix} k,kq,\alpha,\beta;q^2;a^2q/k^2 \\ \gamma,a^2q/k,a^2q^2/k \end{matrix}\right]$$

$$= \sum_{n,r=0}^{\infty} \frac{(k/a;q)_n(k;q)_{n+2r}(\alpha,\beta;q^2)_r(a^2q/k^2)^{n+r}}{(q;q)_n(q^2;q^2)_r(aq;q)_{n+2r}(\gamma;q^2)_r}. \tag{4.5}$$

(vi) Taking $\alpha = \dfrac{a^2q}{k}, \beta = \dfrac{a^2q^2}{k}, \gamma = kq$ in (4.5) we find

$$\sum_{n,r=0}^{\infty} \frac{(k/a;q)_n(k;q)_{n+2r}(a^2q/k, a^2q^2/k;q^2)_r(a^2q/k^2)^{n+r}}{(q;q)_n(q^2;q^2)_r(aq;q)_{n+2r}(kq;q^2)_r}$$

$$= \frac{(aq/k, a^2q/k;q)_\infty(a^2q/k;q^2)_\infty}{(aq, a^2q/k^2;q)_\infty(a^2q/k^2;q^2)_\infty}. \tag{4.6}$$

(vii) Taking $\alpha = \dfrac{a^2q}{k}, \beta = \dfrac{a^2q^2}{k}, \gamma = k$ in (4.5) we get the summation formula;

$$\sum_{n,r=0}^{\infty} \frac{(k/a;q)_n(k;q)_{n+2r}(a^2q/k, a^2q^2/k;q^2)_r(a^2q/k^2)^{n+r}}{(q;q)_n(q^2;q^2)_r(aq;q)_{n+2r}(k;q^2)_r}$$

$$= \frac{(aq/k, a^2q/k;q)_\infty(a^2q^2/k;q^2)_\infty}{(aq, a^2q/k^2;q)_\infty(a^2q/k^2;q^2)_\infty}. \tag{4.7}$$

Similar other results can also be scored.

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