J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.5, No.2 (2016), pp. 09-30

CERTAIN NEW MODULAR EQUATIONS OF MIXED DEGREE IN THE THEORY OF SIGNATURE 3

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Abstract: In this paper, we establish certain new modular equations of mixed degree in the theory of signature 3, which are analogous to the Ramanujan-Russell type modular equation and the Ramanujan-Schläfli type mixed modular equations.

Keywords and Phrases: Theta-function, modular equations.

2000 Mathematics Subject Classification: 11B65, 33D10, 11F27.

1. Introduction

As usual for any complex number a, we define

$$(a)_0 := 1$$

and

$$(a)_n := a(a+1)(a+2)(a+3)...(a+n-1)$$

for any positive integer n. The Gauss hypergeometric series is defined by

$$_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^n, \quad |x| < 1.$$

Suppose that

$$n\frac{{}_2F_1\left[\begin{matrix}\frac{1}{r},\frac{r-1}{r}\\ 1\end{matrix};1-\alpha\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{r},\frac{r-1}{r}\\ 1\end{matrix};\alpha\right]}=\frac{{}_2F_1\left[\begin{matrix}\frac{1}{r},\frac{r-1}{r}\\ 1\end{matrix};1-\beta\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{r},\frac{r-1}{r}\\ 1\end{matrix};\beta\right]}, \tag{1.1}$$

holds for some positive integer n. The modular equation of degree n in signature r is the relation between α, β that is indeed by (1.1). The case $r = 2$ is called classical. S. Ramanujan has recorded many modular equations in his notebooks [13], [14] both in classical theory and alternative theories (r= 3, 4 and 6). A proof of all the modular equations recorded by Ramanujan can be found in [5], [7], [8]. A wonderful introduction of Ramanujan modular equations can be found in [8].

L. Schläfli [16] established certain identities which provides the relation between P and Q, where

$$P = 2^{\frac{1}{6}}[\alpha\beta(1-\alpha)(1-\beta)]^{1/24}$$

and

$$Q = \left[\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right]^{\frac{1}{24}},$$

for β having degrees 3, 5, 7, 11, 13, 17 and 19 respectively over α in the classical theory.

Ramanujan recorded eleven Schläfli type mixed modular equations in his first notebook [13]. R. Russell [15] established certain modular relation which provides the relation between $(\alpha\beta)^{1/8}$ and $((1-\alpha)(1-\beta))^{1/8}$. Ramanujan also recorded certain modular equation of these natures in the theory of signature 3, for details one may refer [7] and [13]. Recently H. H. Chan and W. -C. Liaw [11], M. S. M. Naika [12] and K. R. Vasuki and C. Chamaraju [17] have derived certain new modular equations in the theory of signature 3. In fact, Vasuki and Chamaraju [17] have established certain identities for X, Y, Z and W, where

$$X = 3\left[\frac{(1-\alpha^*)(1-\beta^*)(1-\gamma^*)(1-\delta^*)}{\alpha^*\beta^*\gamma^*\delta^*}\right]^{\frac{1}{12}} \tag{1.2}$$

$$Y = \left[\frac{\alpha^*\beta^*(1-\gamma^*)(1-\delta^*)}{\gamma^*\delta^*(1-\alpha^*)(1-\beta^*)}\right]^{\frac{1}{12}} \tag{1.3}$$

$$Z = \left[\frac{\alpha^*\gamma^*(1-\beta^*)(1-\delta^*)}{\beta^*\delta^*(1-\alpha^*)(1-\gamma^*)}\right]^{\frac{1}{12}} \tag{1.4}$$

and

$$W = \left[\frac{\alpha^*\delta^*(1-\alpha^*)(1-\delta^*)}{\beta^*\gamma^*(1-\beta^*)(1-\gamma^*)}\right]^{\frac{1}{12}}, \tag{1.5}$$

with β^*, γ^* and δ^* having degrees n_1, n_2 and n_1n_2 respectively over α^* in the theory of signature 3.

In Section 2 of this paper, our aim is to establish new modular equations relating X, Y, Z and W. We conclude this introduction, by recalling some definitions and identities which we are going to use in the Section 2. For any complex numbers a and q, with $|q| < 1$, let

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots.$$

In Chapter 16 of his second notebook [1] [13, p. 197] [6, p. 36], Ramanujan define

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty,$$

and

$$\chi(-q) := (q;q^2)_\infty.$$

For convenience, we set $f(-q^n) = f_n$. From [17], we have

$$\frac{f(-q_3)}{q_3^{1/12} f(-q_3^3)} = 3^{\frac{1}{4}}\left[\frac{1-\alpha^*}{\alpha^*}\right]^{\frac{1}{12}}, \tag{1.6}$$

where

$$q_3 = \exp\left[-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{1}{3} \\ 1\end{matrix}; 1-\alpha^*\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{1}{3} \\ 1\end{matrix}; \alpha^*\right]}\right].$$

Using (1.6) in (1.2)-(1.5) respectively and by analytic continuation, we have

$$X = \frac{f_1 f_{n_1} f_{n_2} f_{n_1n_2}}{f_3 f_{3n_1} f_{3n_2} f_{3n_1n_2}}, \tag{1.7}$$

$$Y = \frac{f_{n_2} f_{n_1n_2} f_3 f_{3n_1}}{f_{3n_2} f_{3n_1n_2} f_1 f_{n_1}}, \tag{1.8}$$

$$Z = \frac{f_{n_1} f_{n1n_2} f_3 f_{3n_2}}{f_{3n_1} f_{3n_1n_2} f_1 f_{n_2}}, \tag{1.9}$$

and

$$W = \frac{f_1 f_{n_1 n_2} f_{3n_1} f_{3n_2}}{f_3 f_{3n_1 n_2} f_{n_1} f_{n_2}}. \tag{1.10}$$

In the classical theory, we have from [5, p. 124]

$$\frac{f(q)}{q^{\frac{1}{24}} f(-q^2)} = \frac{2^{\frac{1}{6}}}{[\alpha(1-\alpha)]^{\frac{1}{24}}}, \tag{1.11}$$

where

$$q = \exp\left[-\pi \frac{{}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; 1-\alpha\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \alpha\right]}\right].$$

Let

$$A := [256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)]^{\frac{1}{48}}$$

$$B := \left[\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right]^{1/48}$$

$$C := \left[\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right]^{1/48}$$

and

$$D := \left[\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right]^{1/48}.$$

Then

(i) $$C^6 + \frac{1}{C^6} = D^8 + \frac{1}{D^8} + D^4 + \frac{1}{D^4} - 2, \tag{1.12}$$

where α, β, γ, and δ having degrees 1, 3, 5 and 15 respectively.

(ii) $$B^{12} + \frac{1}{B^{12}} - 18\left(B^6 + \frac{1}{B^6}\right) + 18\sqrt{2}\left(B^3 + \frac{1}{B^3}\right)\left(A^3 + \frac{1}{A^3}\right) - 8\left(A^6 + \frac{1}{A^6}\right) - 54 = 0, \tag{1.13}$$

where α, β, γ, and δ having degrees 1, 3, 7 and 21 respectively.

(iii) $$D^4 + \frac{1}{D^4} - \left(D^2 + \frac{1}{D^2}\right) - 2\left(A^2 + \frac{1}{A^2}\right) = 0, \tag{1.14}$$

where α, β, γ, and δ having degrees 1, 3, 11 and 33 respectively. The modular equation (1.14) is due to Ramanujan, and a simple proof of which has been given by Baruah [4], the modular equation (1.13) is due to Baruah [3], and (1.12) is due to Vasuki and B. R. Srivatsa Kumar [21].

2. Certain new modular equations of mixed degree in the theory of signature 3

In this section, we deduce certain P-Q eta function identities and from them we find certain new modular equation of mixed degree in the theory of signature 3.

Lemma 2.1 Let

$$P := \frac{f_1 f_2}{q^{\frac{1}{4}} f_3 f_6} \qquad \text{and} \qquad Q := \frac{f_2 f_4}{q^{\frac{1}{2}} f_6 f_{12}}.$$

Then,

$$\left(\frac{Q}{P}\right)^8 + \left(\frac{P}{Q}\right)^8 - 7\left\{\left(\frac{Q}{P}\right)^4 + \left(\frac{P}{Q}\right)^4\right\} = \left\{\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2\right\} \times$$
$$\left[(PQ)^2 + \frac{81}{(PQ)^2}\right] + 24.$$

Proof. Let

$$A_n := \frac{f_n}{q^{n/12} f_{3n}}. \tag{2.1}$$

Then, from [13, p. 327], [6, Entry 51, p. 204], we have

$$(A_1A_2)^2 + \frac{9}{(A_1A_2)^2} = \left(\frac{A_2}{A_1}\right)^6 + \left(\frac{A_1}{A_2}\right)^6. \tag{2.2}$$

Changing q to q^2 in (2.2), we obtain

$$(A_2A_4)^2 + \frac{9}{(A_2A_4)^2} = \left(\frac{A_2}{A_4}\right)^6 + \left(\frac{A_4}{A_2}\right)^6.$$

From (2.2) and the above, we obtain

$$(PQ)^2 + \frac{81}{(PQ)^2} + 9\left\{\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2\right\} = \left(\frac{Q}{P}\right)^6 + \left(\frac{P}{Q}\right)^6 +$$
$$\left\{\left(\frac{A_2^2}{A_1A_4}\right)^6 + \left(\frac{A_1A_4}{A_2^2}\right)^6\right\}. \tag{2.3}$$

From [19], we have, if

$$A := q^{\frac{1}{12}} \frac{\chi(q)}{\chi(q^3)} \qquad \text{and} \qquad B := q^{\frac{1}{6}} \frac{\chi(-q^2)}{\chi(-q^6)},$$

then

$$\left[(AB)^2 - \frac{1}{(AB)^2}\right]\left[\left(\frac{A}{B}\right)^6 - \left(\frac{B}{A}\right)^6\right] + (AB)^4 + \frac{1}{(AB)^4} + 6 = 0.$$

Changing q to $-q$ in the above, we obtain

$$\left(\frac{A'}{B}\right)^6 + \left(\frac{B}{A'}\right)^6 = \left[(A'B)^4 + \frac{1}{(A'B)^4} - 6\right]\left[(A'B)^2 + \frac{1}{(A'B)^2}\right]^{-1},$$

where

$$A' := q^{\frac{1}{12}} \frac{\chi(-q)}{\chi(-q^3)}.$$

Now using the fact that

$$\frac{A'}{B} = \frac{A_1A_4}{A_2^2} \text{ and } A'B = \frac{P}{Q},$$

in the above, we find that

$$\left(\frac{A_1A_4}{A_2^2}\right)^6 + \left(\frac{A_2^2}{A_1A_4}\right)^{96} = \left[\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 - 6\right]\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right]^{-1}.$$

Using this in (2.3), we obtain the required result.

Theorem 2.1 If α^*, β^*, γ^* and δ^* have degrees 1, 2, 2 and 4 respectively, then

$$Z^8 + \frac{1}{Z^8} - 7\left[Z^4 + \frac{1}{Z^4}\right] = \left[X^2 + \frac{9^2}{X^2}\right]\left[Z^2 + \frac{1}{Z^2}\right] + 24$$

Proof. The Theorem 2.1 follow from Lemma 2.1, (1.7) and (1.9).

Remark: For a slightly different proof of the Theorem 2.1, one may refer [17].

Lemma 2.2 [13, p. 330], [6, p. 215]. Let

$$P := \frac{f_1f_5}{q^{\frac{1}{2}} f_3f_{15}} \qquad \text{and} \qquad Q := \frac{f_2f_{10}}{qf_6f_{30}}.$$

Then,

$$PQ + \frac{9}{PQ} = \left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 - 4\left[\frac{P}{Q} + \frac{Q}{P}\right]. \tag{2.4}$$

Proof. Changing q to q^5 in (2.2), we obtain

$$(A_5A_{10})^2 + \frac{9}{(A_5A_{10})^2} = \left(\frac{A_5}{A_{10}}\right)^6 + \left(\frac{A_{10}}{A_5}\right)^6.$$

From (2.2) and the above, we deduce that

$$(PQ)^2 + \frac{81}{(PQ)^2} + 9\left\{\left(\frac{A_1A_2}{A_5A_{10}}\right)^2 + \left(\frac{A_5A_{10}}{A_1A_2}\right)^2\right\} =$$

$$\left(\frac{Q}{P}\right)^6 + \left(\frac{P}{Q}\right)^6 + \left(\frac{A_1A_{10}}{A_2A_5}\right)^6 + \left(\frac{A_2A_5}{A_1A_{10}}\right)^6. \tag{2.5}$$

Let

$$B_n := \frac{f_n}{q^{n/24}f_{2n}}. \tag{2.6}$$

Then, from [16], [20], we have

$$(B_1B_3)^3 + \frac{8}{(B_1B_3)^3} = \left(\frac{B_3}{B_1}\right)^6 - \left(\frac{B_1}{B_3}\right)^6. \tag{2.7}$$

Changing q to q^5 in the above, and then multiplying the resulting identity with (2.7), we deduced that

$$(B_1B_3B_5B_{15})^3 + \frac{64}{(B_1B_3B_5B_{15})^3} + 8\left[\left(\frac{B_1B_3}{B_5B_{15}}\right)^3 + \left(\frac{B_5B_{15}}{B_1B_3}\right)^3\right] =$$

$$\left(\frac{Q}{P}\right)^6 + \left(\frac{P}{Q}\right)^6 - \left[\left(\frac{A_1A_{10}}{A_2A_5}\right)^6 + \left(\frac{A_2A_5}{A_1A_{10}}\right)^6\right]. \tag{2.8}$$

From [13, p. 327], [6, p. 205], we have

$$(AB)^2 - \frac{9}{(AB)^2} = \left(\frac{B}{A}\right)^3 - 8\left(\frac{A}{B}\right)^3, \tag{2.9}$$

where

$$A := \frac{f_2}{q^{\frac{1}{24}}f_3} \qquad \text{and} \qquad B := \frac{f_1}{q^{\frac{5}{24}}f_6}.$$

Changing q to q^5 in the above and then multiplying the resulting identity with (2.9), we obtain

$$(PQ)^2+\frac{81}{(PQ)^2}-9\left\{\left(\frac{A_1A_2}{A_5A_{10}}\right)^2+\left(\frac{A_5A_{10}}{A_1A_2}\right)^2\right\}=$$

$$(B_1B_2B_5B_{15})^3+\frac{64}{(B_1B_2B_5B_{15})^3}-8\left[\left(\frac{B_1B_3}{B_5B_{15}}\right)^3+\left(\frac{B_5B_{15}}{B_1B_3}\right)^3\right]. \qquad (2.10)$$

From (2.10) and (2.8), we deduce that

$$(PQ)^2+\frac{81}{(PQ)^2}-9\left\{\left(\frac{A_1A_2}{A_5A_{10}}\right)^2+\left(\frac{A_5A_{10}}{A_1A_2}\right)^2\right\}+16\left[\left(\frac{B_1B_3}{B_5B_{15}}\right)^3+\left(\frac{B_5B_{15}}{B_1B_3}\right)^3\right]=$$

$$\left(\frac{P}{Q}\right)^6+\left(\frac{Q}{P}\right)^6-\left\{\left(\frac{A_1A_{10}}{A_2A_5}\right)^6+\left(\frac{A_2A_{15}}{A_1A_{10}}\right)^6\right\}.$$

From (2.5) and the above, we found that

$$(PQ)^2+\frac{81}{(PQ)^2}+8\left[\left(\frac{B_1B_3}{B_5B_{15}}\right)^3+\left(\frac{B_5B_{15}}{B_1B_3}\right)^3\right]=\left(\frac{P}{Q}\right)^6+\left(\frac{Q}{P}\right)^6. \qquad (2.11)$$

Employing (1.11) in (1.12), we deduce that

$$\left(\frac{B_1B_3}{B_5B_{15}}\right)^3+\left(\frac{B_5B_{15}}{B_1B_3}\right)^3=\left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4-\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]-2.$$

Using this in (2.11), we obtain

$$(PQ)^2+\frac{81}{(PQ)^2}=\left(\frac{P}{Q}\right)^6+\left(\frac{Q}{P}\right)^6-8\left[\left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4\right]+8\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]+16.$$

This implies

$$\left(PQ+\frac{9}{PQ}\right)^2=\left[\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3-4\left(\frac{P}{Q}+\frac{Q}{P}\right)\right]^2.$$

Taking square root on both sides of the above, we complete the proof.

Theorem 2.2 If α^*, β^*, γ^* and δ^* have degrees 1, 2, 5 and 10 respectively, then

$$X + \frac{9}{X} = Z^3 + \frac{1}{Z^3} - 4\left[Z + \frac{1}{Z}\right].$$

Proof. From (1.7) and (1.9), we have

$$PQ = X \qquad \text{and} \qquad \frac{P}{Q} = Z,$$

where P and Q are as in Lemma 2.2. Using these in Lemma 2.2, we obtain the required result.

Lemma 2.3 [13, p. 330], [6, p. 218] If

$$P := \frac{f_6 f_5}{q^{\frac{1}{4}} f_2 f_{15}} \qquad \text{and} \qquad Q := \frac{f_3 f_{10}}{q^{\frac{3}{4}} f_1 f_{36}},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - 1.$$

Recently Bhargava, Vasuki and Rajanna [10] have proved Lemma 2.3 using only theta function identities which are deduced from Ramanujan's ${}_1\psi_1$ summation formula.

Theorem 2.3 If α^*, β^*, γ^* and δ^* have degrees 1, 2, 5 and 10 respectively, then

$$Y + \frac{1}{Y} = Z^2 + \frac{1}{Z^2} - 1.$$

Proof. We have, from (1.8) and (1.9)

$$PQ = Y \qquad \text{and} \qquad \frac{Q}{P} = Z,$$

where P and Q are as in Lemma 2.3. Using these in Lemma 2.3, we obtain the required result.

Lemma 2.4 [20] If

$$P := \frac{f_1 f_2}{q^{\frac{1}{4}} f_3 f_6} \qquad \text{and} \qquad Q := \frac{f_5 f_{10}}{q^{\frac{5}{4}} f_{15} f_{30}},$$

then,

$$(PQ)^2 + \frac{81}{(PQ)^2} = \left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 - 5\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right] - 5\left[\frac{P}{Q} + \frac{Q}{P}\right] + 20.$$

For a proof, see [20].

Theorem 2.4 If α^*, β^*, γ^* and δ^* have degrees 1, 2, 5, and 10 respectively, then

$$X^2 + \frac{81}{X^2} = Y^3 + \frac{1}{Y^3} - 5\left[Y^2 + \frac{1}{Y^2}\right] - 5\left[Y + \frac{1}{Y}\right] + 20.$$

Proof. We have, from (1.7) and (1.8),

$$PQ = X \qquad \text{and} \qquad \frac{Q}{P} = Y,$$

where P and Q are as in Lemma 2.4. Using these in Lemma 2.4, we obtain the Theorem 2.4.

Lemma 2.5 [13, p. 330], [6, p. 214]. If

$$P := \frac{f_3 f_5}{q^{\frac{1}{3}} f_1 f_{15}} \qquad \text{and} \qquad Q := \frac{f_6 f_{10}}{q^{\frac{2}{3}} f_2 f_{30}},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 + 4.$$

For a simple proof of the above using Ramanujan's ${}_1\psi_1$ summation formula, see [10].

Theorem 2.5 If α^*, β^*, γ^* and δ^* have degrees 1, 2, 5 and 10 respectively, then

$$W^3 + \frac{1}{W^3} + 4 = Y + \frac{1}{Y}.$$

Proof. We have, from (1.8) and (1.10),

$$PQ = W \qquad \text{and} \qquad \frac{Q}{P} = Y,$$

where P and Q are as in Lemma 2.5. Using these in Lemma 2.5, we obtain the Theorem 2.5.

Lemma 2.6 If

$$P := \frac{f_1 f_7}{q^{\frac{2}{3}} f_3 f_{21}} \qquad \text{and} \qquad Q := \frac{f_2 f_{14}}{q^{\frac{4}{3}} f_6 f_{42}},$$

then

$$\left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6 + (PQ)^2 + \frac{81}{(PQ)^2} + 4\left[PQ + \frac{9}{PQ}\right] + 20 =$$
$$2\left[\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3\right]\left[10 + PQ + \frac{9}{PQ}\right].$$

The above Lemma is due to Baruah [2]. A simple proof of the same have been given by Vasuki and Sharath [18].

Theorem 2.6 Let α^*, β^*, γ^* and δ^* have degrees 1, 2, 7 and 14 respectively. Then,

$$Z^6 + \frac{1}{Z^6} + X^2 + \frac{81}{X^2} + 4\left[X + \frac{9}{X}\right] + 20 = 2\left[Z^3 + \frac{1}{Z^3}\right]\left[10 + X + \frac{9}{X}\right].$$

Proof. It is easy to see from (1.7) and (1.9), that

$$PQ = X \qquad \text{and} \qquad \frac{Q}{P} = Z,$$

where P and Q are as in Lemma 2.6. Using these in Lemma 2.6, we obtain the required result.

Lemma 2.7 If

$$P := q^{\frac{13}{12}} \frac{f_1 f_{42}}{f_3 f_{14}} \qquad \text{and} \qquad Q := \frac{f_6 f_7}{q^{\frac{5}{12}} f_2 f_{21}},$$

then

$$(PQ)^9 + \frac{1}{(PQ)^9} - 2\left[(PQ)^6 + \frac{1}{(PQ)^6}\right] - 7\left[(PQ)^3 + \frac{1}{(PQ)^3}\right] =$$
$$\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 + \left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right]\left[5\left((PQ)^3 + \frac{1}{(PQ)^3}\right) + 6\right] + 14.$$

Proof. Changing q to q^7 in (2.2) then multiplying the resulting identity with (2.2), we find that

$$(A_1 A_2 A_7 A_{14})^2 + \frac{81}{(A_1 A_2 A_7 A_{14})^2} + 9\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right] =$$

$$(PQ)^6+\frac{1}{(PQ)^6}+\left(\frac{A_2A_7}{A_1A_{14}}\right)^6+\left(\frac{A_1A_{14}}{A_2A_7}\right)^6. \tag{2.12}$$

Changing q to q^7 in (2.7) and then multiplying the resulting identity with (2.7), we find that

$$(B_1B_3B_7B_{21})^3+\frac{64}{(B_1B_3B_7B_{21})^2}+8\left[\left(\frac{B_1B_3}{B_7B_{21}}\right)^3+\left(\frac{B_7B_{21}}{B_1B_3}\right)^3\right]=$$

$$(PQ)^6+\frac{1}{(PQ)^6}-\left[\left(\frac{A_1A_{14}}{A_2A_7}\right)^6+\left(\frac{A_2A_7}{A_1A_{14}}\right)^6\right]. \tag{2.13}$$

Changing q to q^7 in (2.9) and then multiplying the resulting identity with (2.9), we obtain

$$(A_1A_2A_7A_{14})^2+\frac{81}{(A_1A_2A_7A_{14})^2}-9\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]=$$

$$(B_1B_3B_7B_{21})^3+\frac{64}{(B_1B_3B_7B_{21})^2}-8\left[\left(\frac{B_1B_3}{B_7B_{21}}\right)^3+\left(\frac{B_7B_{21}}{B_1B_3}\right)^3\right]. \tag{2.14}$$

Subtracting (2.12) from (2.14), we obtain

$$(B_1B_3B_7B_{21})^3+\frac{64}{(B_1B_3B_7B_{21})^3}+18\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]=$$

$$(PQ)^6+\frac{1}{(PQ)^6}+\left(\frac{A_2A_7}{A_1A_{14}}\right)^6+\left(\frac{A_1A_{14}}{A_2A_7}\right)^6+8\left[\left(\frac{B_1B_3}{B_7B_{21}}\right)^3+\left(\frac{B_7B_{21}}{B_1B_3}\right)^3\right].$$

Adding the above with (2.13), we obtain

$$(B_1B_3B_7B_{21})^3+\frac{64}{(B_1B_3B_7B_{21})^3}+9\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]=(PQ)^6+\frac{1}{(PQ)^6}. \tag{2.15}$$

Employing (1.11) in (1.13), we obtain

$$(PQ)^6+\frac{1}{(PQ)^6}-18\left[(PQ)^3+\frac{1}{(PQ)^3}\right]+9\left[(PQ)^{\frac{3}{2}}+\frac{1}{(PQ)^{\frac{3}{2}}}\right]\left[X_1^{\frac{3}{2}}+\frac{8}{X_1^{\frac{3}{2}}}\right]-$$

$$\left[X_1^3 + \frac{64}{X_1^3}\right] - 54 = 0,$$

where $X_1 = B_1B_3B_7B_{21}$. Using (2.15) in the above, we find that

$$\left[(PQ)^{\frac{3}{2}} + \frac{1}{(PQ)^{\frac{3}{2}}}\right]\left[X^{\frac{3}{2}} + \frac{8}{X^{\frac{3}{2}}}\right] = 2\left[(PQ)^3 + \frac{1}{(PQ)^3}\right] - \left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right] + 6.$$

Squaring this on both sides and then using (2.15), to eliminate X, we obtain the required result.

Theorem 2.7 Let α^*, β^*, γ^* and δ^* have degrees 1, 2, 7 and 14 respectively. Then,

$$Z^9 + \frac{1}{Z^9} - 2\left[Z^6 + \frac{1}{Z^6}\right] - 7\left[Z^3 + \frac{1}{Z^3}\right] = Y^4 + \frac{1}{Y^4} +$$

$$\left[Y^2 + \frac{1}{Y^2}\right]\left[5\left(Z^3 + \frac{1}{Z^3}\right) + 6\right] + 14.$$

Proof. It is easy to see from (1.8) and (1.9) that

$$PQ = Z \qquad \text{and} \qquad \frac{P}{Q} = Y,$$

where P and Q are as in Lemma 2.7. Using these in Lemma 2.7, we obtain the required result.

Lemma 2.8 Let

$$P := \frac{f_1f_2}{q^{\frac{1}{4}}f_3f_6} \qquad \text{and} \qquad Q := \frac{f_7f_{14}}{q^{\frac{7}{4}}f_{21}f_{42}}.$$

Then,

$$\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 - 42\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right] - 7\left[PQ + \frac{9}{PQ}\right]\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right]$$

$$= (PQ)^3 + \frac{9^3}{(PQ)^3} + 14\left[(PQ)^2 + \frac{9^2}{(PQ)^2}\right] + 105\left[PQ + \frac{9}{PQ}\right] + 434.$$

Proof. Let

$$C := \frac{f_1f_6f_7f_{42}}{q^{\frac{2}{3}}f_2f_3f_{14}f_{21}}. \qquad M := \frac{q^{\frac{1}{4}}f_3f_6}{f_1f_2} \qquad \text{and} \qquad N := \frac{q^{\frac{7}{4}}f_{21}f_{42}}{f_7f_{14}}$$

Then, from Lemma 2.7, we have

$$C^9+\frac{1}{C^9}-2\left[C^6+\frac{1}{C^6}\right]-7\left[C^3+\frac{1}{C^3}\right]=\left(\frac{N}{M}\right)^4+$$

$$\left(\frac{M}{N}\right)^4+\left[\left(\frac{N}{M}\right)^2+\left(\frac{M}{N}\right)^2\right]\left[5\left(C^3+\frac{1}{C^3}\right)+6\right]+14, \tag{2.16}$$

and from Lemma 2.6, we have

$$C^6+\frac{1}{C^6}+\frac{1}{(MN)^2}+81(MN)^2+4\left[\frac{1}{MN}+9MN\right]+20=$$

$$2\left[C^3+\frac{1}{C^3}\right]\left[\frac{1}{MN}+9MN+10\right]. \tag{2.17}$$

Eliminating C between (2.16) and (2.17) using Maple, we obtain

$$A(M,N)B(M,N)=0,$$

where

$$A(M,N)=N^8+M^8-MN-63M^7N^3-63M^3N^7-7MN^5-7M^5N-$$
$$434(MN)^4-42M^6N^2-42M^2N^6-14(MN)^3-945(MN)^5-105(MN)^3-$$
$$729(MN)^7-1134(MN)^6,$$

and

$$B(M,N)=N^8+M^8-MN+153M^7N^3+153M^3N^7+17MN^5+17M^5N+$$
$$414(MN)^4+154M^6N^2+154M^2N^6+6(MN)^2+63(MN)^5+7(MN)^3-$$
$$729(MN)^7+486(MN)^6.$$

By definition of M and N, we see that

$$M=q^{\frac{1}{4}}\left(1+q+3q^2+3q^3+8q^4+9q^5+...\right)$$

and

$$N=q^{\frac{7}{4}}\left(1+q^7+3q^{14}+3q^{21}+...\right).$$

Using these in $A(M,N)$ and $B(M,N)$, we find that

$$A(M,N)=-588q^{13}-2681q^{14}-17969q^{15}-728289q^{16}+...,$$

and

$$B(M,N) = 24q^3 + 140q^4 + 836q^5 + 3468q^6 + ...,$$

Now $q^{-3}B(M,N) \neq 0$ as $q \to 0$, where as $q^{-3}A(M,N) \to 0$ as $q \to 0$ thus $q^{-3}A(M,N) = 0$ in some neighborhood of $q = 0$. Thus by analytic continuation, $A(M,N) = 0$ for all values q with $|q| < 1$. Using the fact that $M = \frac{1}{P}$ and $N = \frac{1}{Q}$ in $B(M,N) = 0$, we obtain the required result.

Theorem 2.8 Let α^*, β^*, γ^* and δ^* have degrees 1, 2, 7 and 14 respectively. Then,

$$Y^4 + \frac{1}{Y^4} - 42\left[Y^2 + \frac{1}{Y^2}\right] - 7\left[X + \frac{9}{X}\right]\left[Y^2 + \frac{1}{Y^2}\right] =$$

$$X^3 + \frac{9^3}{X^3} + 14\left[X^2 + \frac{9^2}{X^2}\right] + 105\left[X + \frac{9}{X}\right] + 434.$$

Proof. We have, from (1.7) and (1.8), that

$$PQ = X \qquad \text{and} \qquad \frac{Q}{P} = Y,$$

where P and Q are as in Lemma 2.8. Using these in Lemma 2.8, we obtain the required result.

Lemma 2.9 Let

$$P := \frac{f_1 f_2}{q^{\frac{1}{4}} f_3 f_6} \qquad \text{and} \qquad Q := \frac{f_9 f_{18}}{q^{\frac{9}{4}} f_{27} f_{54}}.$$

Then,

$$(PQ)^4 + \frac{9^4}{(PQ)^4} + \left[(PQ)^2 + \frac{9^2}{(PQ)^2}\right]\left[9\frac{Q^2}{P^2} + 27\frac{P}{Q} - 51\frac{Q}{P} + 27\right] + 81 =$$

$$\left(\frac{Q}{P}\right)^5 - 27\left(\frac{Q}{P}\right)^4 + 225\left(\frac{Q}{P}\right)^3 - 9\left[73\left(\frac{Q}{P}\right)^2 + 81\left(\frac{P}{Q}\right)^2\right] + 81\left[5\frac{Q}{P} + 9\frac{P}{Q}\right].$$

Proof. Let

$$R := \frac{f_3 f_6}{q^{\frac{3}{4}} f_9 f_{18}}.$$

Then, from [17], we have

$$\left(\frac{R}{P}\right)^2 = PR + \frac{9}{PR} + 3. \tag{2.18}$$

Changing q to q^3 in the above, we obtain

$$\left(\frac{Q}{R}\right)^2 = QR + \frac{9}{QR} + 3.$$

Now eliminating R between (2.18) and the above, we obtain the required result.

Theorem 2.9 Let α^*, β^*, γ^* and δ^* have degrees 1, 2, 9 and 18 respectively. Then,

$$X^4 + \frac{9^4}{X^4} + \left[X^2 + \frac{9^2}{X^2}\right]\left[9Y^2 + 27\frac{1}{Y} - 51Y + 27\right] + 81 =$$

$$Y^5 - 27Y^4 + 225Y^3 - 9\left[73Y^2 + 81\frac{1}{Y^2}\right] + 81\left[5Y + \frac{9}{Y}\right].$$

Proof. We have, from (1.7) and (1.8), that

$$PQ = X \qquad \text{and} \qquad \frac{Q}{P} = Y.$$

where P and Q are as in Lemma 2.9. Using these in Lemma 2.9, we obtain the required result.

Lemma 2.10 [9] Let

$$P := \frac{f_1 f_{11}}{q f_3 f_{33}} \qquad \text{and} \qquad Q := \frac{f_2 f_{22}}{q^2 f_6 f_{66}}.$$

Then,

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 - 4\left[\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2\right] + 4.$$

Proof. Multiplying Entry 14(i) and 14(ii) [5, p. 408] to eliminate $\sqrt{mm'}$ and then transforming the resulting modular equation in theta function identity, we obtain Lemma 2.10.

Theorem 2.10 Let α^*, β^*, γ^* and δ^* have degrees 1, 2, 11 and 22 respectively. Then,

$$X + \frac{9}{X} = Z^3 + \frac{1}{Z^3} - 4\left[Z^2 + \frac{1}{Z^2}\right] + 4.$$

Proof. We have, from (1.7) and (1.9) that

$$PQ = X \qquad \text{and} \qquad \frac{P}{Q} = Z,$$

where P and Q are as in Lemma 2.10. Using these in Lemma 2.10, we obtain the required result.

Lemma 2.11 Let

$$P := q^{\frac{7}{4}} \frac{f_1 f_{66}}{f_2 f_{22}} \qquad \text{and} \qquad Q := \frac{f_6 f_{11}}{q^{\frac{3}{4}} f_2 f_{33}}.$$

Then,

$$(PQ)^5 + \frac{1}{(PQ)^5} - 3\left[(PQ)^4 + \frac{1}{(PQ)^4}\right] + 4\left[(PQ)^3 + \frac{1}{(PQ)^3}\right] - 5\left[(PQ)^2 + \frac{1}{(PQ)^2}\right] +$$
$$7\left[PQ + \frac{1}{PQ}\right] = \left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 6.$$

Proof. Changing q to q^{11} in (2.2) then multiplying the resulting identity with (2.2), we find that

$$(A_1A_2A_{11}A_{22})^2 + \frac{81}{(A_1A_2A_{11}A_{22})^2} + 9\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right] =$$
$$(PQ)^6 + \frac{1}{(PQ)^6} + \left(\frac{A_2A_{11}}{A_1A_{22}}\right)^6 + \left(\frac{A_1A_{22}}{A_2A_{11}}\right)^6. \tag{2.19}$$

Changing q to q^{11} in (2.7) and then multiplying the resulting identity with (2.7), we obtain

$$(B_1B_3B_{11}B_{33})^3 + \frac{64}{(B_1B_3B_{11}B_{33})^3} + 8\left[\left(\frac{B_1B_3}{B_{11}B_{33}}\right)^3 + \left(\frac{B_{11}B_{33}}{B_1B_3}\right)^3\right] =$$
$$(PQ)^6 + \frac{1}{(PQ)^6} - \left[\left(\frac{A_1A_{22}}{A_2A_{11}}\right)^6 + \left(\frac{A_2A_{11}}{A_1A_{22}}\right)^6\right]. \tag{2.20}$$

Changing q to q^{11} in (2.9) and then multiplying the same with (2.9), we obtain

$$(A_1A_2A_{11}A_{22})^2 + \frac{81}{(A_1A_2A_{11}A_{22})^2} - 9\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right] =$$
$$(B_1B_3B_{11}B_{33})^3 + \frac{64}{(B_1B_3B_{11}B_{33})^3} - 8\left[\left(\frac{B_1B_3}{B_{11}B_{33}}\right)^3 + \left(\frac{B_{11}B_{33}}{B_1B_3}\right)^3\right]. \tag{2.21}$$

Now adding (2.19) and (2.20) and then subtracting the resulting identity from (2.21), we deduce that

$$9\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]+8\left[\frac{(B_1B_3B_{11}B_{33})^3}{8}+\frac{8}{(B_1B_3B_{11}B_{33})^3}\right]=$$

$$(PQ)^6+\frac{1}{(PQ)^6}. \tag{2.22}$$

Employing (1.11) in (1.14), we deduce that

$$(PQ)^2+\frac{1}{(PQ)^2}-3\left[PQ+\frac{1}{PQ}\right]-2\left[\frac{X}{2}+\frac{2}{X}\right]=0,$$

where

$$X=\frac{1}{B_1B_3B_{11}B_{33}}.$$

Now, eliminating X between (2.22) and the above, we obtain the required result.

Theorem 2.11 Let α^*, β^*, γ^* and δ^* have degrees 1, 2, 11 and 22 respectively. Then,

$$Z^5+\frac{1}{Z^5}-3\left[Z^4+\frac{1}{Z^4}\right]+4\left[Z^3+\frac{1}{Z^3}\right]-5\left[Z^2+\frac{1}{Z^2}\right]+7\left[Z+\frac{1}{Z}\right]=Y^2+\frac{1}{Y^2}+6.$$

Proof. We have, from (1.8) and (1.9) that

$$PQ=Z \qquad \text{and} \qquad \frac{Q}{P}=Y,$$

where P and Q are as in Lemma 2.11. Using these in Lemma 2.11, we obtain the required result.

Lemma 2.12 Let

$$P:=\frac{f_1f_2}{q^{\frac{1}{4}}f_3f_6} \qquad \text{and} \qquad Q:=\frac{f_{11}f_{22}}{q^{\frac{11}{4}}f_{33}f_{66}}.$$

Then,

$$(PQ)^5+\frac{9^5}{(PQ)^5}-22\left[(PQ)^4+\frac{9^4}{(PQ)^4}\right]+143\left[(PQ)^3+\frac{9^3}{(PQ)^3}\right]-$$

$$396\left[(PQ)^2+\frac{9^2}{(PQ)^2}\right]+2992\left[PQ+\frac{9}{PQ}\right]=\left(\frac{P}{Q}\right)^6+\left(\frac{Q}{P}\right)^6-\left[\left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4\right]$$

$$\left[286+44\left(PQ+\frac{9}{PQ}\right)\right]-\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]\left[2893-275\left(PQ+\frac{9}{PQ}\right)\right.$$

$$\left.-110\left((PQ)^2+\frac{9^2}{(PQ)^2}\right)+11\left((PQ)^3+\frac{9^3}{(PQ)^3}\right)\right]+16280.$$

The proof of Lemma 2.12 is same as proof of Lemma 2.8. We use Lemma 2.10 and Lemma 2.11 to prove Lemma 2.12.

Theorem 2.12 Let α^*, β^*, γ^* and δ^* have degrees 1, 2, 11 and 22 respectively. Then,

$$X^5+\frac{9^5}{X^5}-22\left[X^4+\frac{9^4}{X^4}\right]+143\left[X^3+\frac{9^3}{X^3}\right]-396\left[X^2+\frac{9^2}{X^2}\right]+2992\left[X+\frac{9}{X}\right]=$$

$$\frac{1}{Y^6}+Y^6-\left[\frac{1}{Y^4}+Y^4\right]\left[286+44\left(X+\frac{9}{X}\right)\right]-\left[\frac{1}{Y^2}+Y^2\right]\left[2893-275\left(X+\frac{9}{X}\right)\right.$$

$$\left.-110\left(X^2+\frac{9^2}{X^2}\right)+11\left(X^3+\frac{9^3}{X^3}\right)\right]+16280.$$

Proof. We have, from (1.7) and (1.8) that

$$PQ=X \qquad \text{and} \qquad \frac{Q}{P}=Y,$$

where P and Q are as in Lemma 2.12. Using these in Lemma 2.12, we obtain the required result.

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