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COMPUTABLE REPRESENTATION OF ULTRA GAMMA INTEGRAL

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Abstract: A certain integral is there in the literature which some authors call ultra gamma function, some others call it generalized gamma, some others call it Krätzel integral, some others call it inverse Gaussian integral, some others call it reaction-rate probability integral, some others call it Bessel integral, some others call it the unconditional density in a Bayesian structure and some others call it the Mellin convolution of a product. Thus, this integral is very important to various people in different disciplines. In this article, this integral is evaluated in computable series form. It is shown that the names, generalized gamma and ultra gamma are not appropriate for this integral.

Keywords and Phrases: Mellin convolution, Krätzel integral, ultra gamma function, generalized gamma function, Bessel integral, reaction-rate probability integral, inverse Gaussian integral, computable series form.

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1. Introduction

Consider the integral

$$B = \int_0^\infty x^{\gamma-1} \mathrm{e}^{-ax^\delta - bx^{-\rho}} \mathrm{d}x \tag{1.1}$$

for $a > 0, b > 0, \gamma > 0, \delta > 0, \rho > 0$. If the integrand in B is to be made a statistical density then we may multiply the integral by the normalizing constant.

In that case the function is defined for $x \geq 0$ and zero otherwise. The integrand in B for $\delta = 1, \rho = 1$ and multiplied by the normalizing constant is the inverse Gaussian density for appropriate values of a, b, γ. For $\delta = 1, \rho = \frac{1}{2}$ it is the basic reaction-rate probability integral in nuclear reaction-rate theory. For general values of δ and ρ, Mathai and Haubold (1988) call the integral the generalized reaction-rate probability integral. For the general parameters case, there is no physical interpretation yet but the theory is worked out in Mathai and Haubold (1988) and in the later papers. For $\delta = 1, \rho = 1$ the integral is the basic Krätzel integral in applied analysis, which is also connected to Krätzel transform, see Krätzel (1979), Mathai (2012). Hence one may call (1.1) as the generalized Krätzel integral. If $b = 0$ then it is a generalized gamma integral but when $b = 0$ the special nature of (1.1) is lost. Hence it is not appropriate to call (1.1) as generalized gamma or ultra gamma function because the connection to gamma function is only when $b = 0$ and this is not an admissible value in (1.1). All sorts of studies are done by people working in special functions, treating (1.1) as generalization of gamma function. The Mellin convolution of a product has the structure

$$g(u) = \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v \tag{1.2}$$

so that the Mellin transform of g, with the Mellin parameter s, denoted by $M_g(s)$, is given by

$$M_g(s) = M_{f_1}(s) M_{f_2}(s). \tag{1.3}$$

This is the Mellin convolution of a product property. Now, let

$$f_1(x_1) = \mathrm{e}^{-x_1^{\rho}}, 0 \leq x_1 < \infty, \rho > 0, u = b^{\frac{1}{\rho}} \tag{1.4}$$

$$f_2(x_2) = x_2^{\gamma} \mathrm{e}^{-a x_2^{\delta}}, 0 \leq x_2 < \infty, \delta > 0, \gamma > 0. \tag{1.5}$$

Then

$$\int_0^{\infty} \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v = \int_0^{\infty} x^{\gamma-1} \mathrm{e}^{-a x^{\delta} - b x^{-\rho}} \mathrm{d}v \tag{1.6}$$

where $b = u^{\frac{1}{\rho}}$. Thus, (1.1) is a Mellin convolution of a product or it is also the statistical density of a product of the form $u = x_1 x_2$ where x_1 and x_2 enjoy product probability property (PPP) or are statistically independently distributed positive real scalar random variables. In this case, multiply (1.1) with the normalizing constant. A structure of the type in (1.1) is also of interest for statisticians working on different topics. Connections to inverse Gaussian density and density of a product are already pointed out. It is also of interest for people working in Bayesian analysis. Consider a conditional density of a real scalar positive random variable y, at given value of a parameter or another variable x, which is a generalized gamma density of the following form:

$$f(y|x) = c_1 \mathrm{e}^{-\frac{y}{x^\rho}}, 0 \le y < \infty, x > 0 \tag{1.7}$$

and zero elsewhere, where c_1 is the normalizing constant. Consider the marginal density of x, a generalized gamma density of the form:

$$g(x) = c_2 x^{\gamma-1} \mathrm{e}^{-ax^\delta}, 0 < x < \infty, a > 0, \gamma > 0 \tag{1.8}$$

and zero elsewhere, where c_2 is the normalizing constant. Then the unconditional density of y, denoted by $B(y)$, is available by integrating over the density $g(x)$. That is,

$$B(y) = c_1 c_2 \int_0^\infty x^{\gamma-1} \mathrm{e}^{-ax^\delta - yx^{-\rho}} \mathrm{d}x \tag{1.9}$$

which is nothing but (1.1), multiplied by the constant $c_1 c_2$. Hence, from the point of view of Bayesian analysis also the integral in (1.1) is very important. The integral in (1.1) can be interpreted as a continuous mixture in statistical distribution theory. Since it is a very interesting integral in many topics, we will evaluate it explicitly and represent it in computable forms.

2. Evaluation of the Bessel Integral

In B of (1.1) let us take the Mellin transform with respect to b, with Mellin parameter s, denoted by $M_b(s)$, or take it as the Mellin transform of the function $B(y)$ of (1.9) with $b = y$. Then

$$M_b(s) = \int_0^\infty b^{s-1} [\int_0^\infty x^{\gamma-1} \mathrm{e}^{-ax^\delta - bx^{-\rho}} \mathrm{d}x] \mathrm{d}b. \tag{2.1}$$

Interchange of integrals is valid here and taking the integral over b and then integral over x we have the following:

$$\int_{b=0}^\infty b^{s-1} \mathrm{e}^{-bx^{-\rho}} \mathrm{d}b = \Gamma(s)(x^{-\rho})^{-s} = x^{\rho s}\Gamma(s), \Re(s) > 0. \tag{2.2}$$

Now, the integral over x gives the following:

$$\int_0^\infty x^{\gamma+\rho s-1} \mathrm{e}^{-ax^\delta} \mathrm{d}x = \frac{1}{\delta}\Gamma(\frac{\gamma+\rho s}{\delta}) a^{-(\frac{\gamma+\rho s}{\delta})}. \tag{2.3}$$

Therefore, from (2.2) and (2.3), we have

$$M_b(s) = \frac{1}{\delta a^{\frac{\gamma}{\delta}}}\Gamma(s)\Gamma(\frac{\gamma}{\delta}+\frac{\rho}{\delta}s)a^{-\frac{\rho}{\delta}s}. \tag{2.4}$$

Hence by taking the inverse Mellin transform of (2.4) we get B as the inverse Mellin transform. That is,

$$B = \frac{1}{\delta a^{\frac{\gamma}{\delta}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(\frac{\gamma}{\delta}+\frac{\rho}{\delta}s)(ba^{\frac{\rho}{\delta}})^{-s}\mathrm{d}s, i=\sqrt{-1} \tag{2.5}$$

where the c in the contour is any positive number. The integral in (2.5) can be written as a H-function, for the theory and applications of H-function, see for example Mathai et al. (2010). That is

$$B = \frac{1}{\delta z^{\frac{\gamma}{\delta}}}H_{0,2}^{2,0}[ba^{\frac{\rho}{\delta}}|_{(0,1),(\frac{\gamma}{\delta},\frac{\rho}{\delta})}]. \tag{2.6}$$

When $\rho=\delta$ the coefficient of s in the inverse Mellin transform in (2.5) is 1 and hence one can write it as a G-function. For the theory and applications of G-function, see for example, Mathai (1993). That is, for $\rho=\delta$,

$$B = (\delta a^{\frac{\gamma}{\delta}})^{-1}G_{0,2}^{2,0}[ab|_{0,\frac{\gamma}{\delta}}]. \tag{2.7}$$

Series Representations

The poles of $\Gamma(s)$ are at $s=0,-1,...$ and the poles of $\Gamma(\frac{\gamma}{\delta}+\frac{\rho}{\delta}s)$ are $\frac{\gamma}{\delta}+\frac{\rho}{\delta}s=-\nu,\nu=0,1,2,...$ or $s=-\frac{\gamma}{\rho}-\frac{\delta}{\rho}\nu,\nu=0,1,2,....$ Hence if $\frac{\gamma}{\rho}+\frac{\delta}{\rho}\nu,\nu=0,1,2,...$ is not a positive integer then the poles of the integrand in the Mellin-Barnes representation in (2.5) are simple and then evaluating the sums of residues at the poles of $\Gamma(s)$ and $\Gamma(\frac{\gamma}{\delta}+\frac{\rho}{\delta}s)$ we have the explicit series form. Sum of the residues at the poles of $\Gamma(s), s=-\nu,\nu=0,1,2,...$ is

$$(\delta a^{\frac{\gamma}{\delta}})^{-1}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\frac{\gamma}{\delta}-\frac{\rho}{\delta}\nu)(ba^{\frac{\rho}{\delta}})^{\nu}. \tag{i}$$

For computing the sum of residues at the poles of $\Gamma(\frac{\gamma}{\delta}+\frac{\rho}{\delta}s)$ it is convenient to make a transformation $\frac{\gamma}{\delta}+\frac{\rho}{\delta}s=s_1 \Rightarrow s=-\frac{\gamma}{\rho}+\frac{\delta}{\rho}s_1, \mathrm{d}s=\frac{\delta}{\rho}\mathrm{d}s_1$,

$$(ba^{\frac{\rho}{\delta}})^{-s} = (ba^{\frac{\rho}{\delta}})^{\frac{\gamma}{\rho}-\frac{\delta}{\rho}s_1} = b^{\frac{\gamma}{\rho}}a^{\frac{\gamma}{\delta}}(ba)^{-s_1}.$$

and the sum of the residues is the following:

$$\frac{b^{\frac{\gamma}{\rho}}}{\rho}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(-\frac{\gamma}{\rho}-\frac{\delta}{\rho}\nu)(ab)^{\nu}. \qquad (ii)$$

Therefore for the simple poles case, B is available as the sum of (i) and (ii). That is,

$$\begin{aligned} B &= (\delta a^{\frac{\gamma}{\delta}})^{-1}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\frac{\gamma}{\delta}-\frac{\rho}{\delta}\nu)(ba^{\frac{\rho}{\delta}})^{\nu} \\ &+ \frac{b^{\frac{\gamma}{\rho}}}{\rho}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(-\frac{\gamma}{\rho}-\frac{\delta}{\rho}\nu)(ab)^{\nu}, \end{aligned} \qquad (2.8)$$

for $\frac{\gamma}{\rho}+\frac{\delta}{\rho}\nu$ is not a positive integer for $\nu = 0, 1, ...,$ $\frac{\gamma}{\delta}-\frac{\rho}{\delta}\nu \neq 0, -1, -2, ...$ for $\nu = 0, 1, ...$ For $\rho = \delta$ one can simplify in terms of ${}_0F_1$ series.

Special case (1): $\rho = \delta$ **and** $\frac{\gamma}{\delta}$ **is not a positive integer**

Then

$$\Gamma(\frac{\gamma}{\delta}) = (\frac{\gamma}{\delta}-1)(\frac{\gamma}{\delta}-2)...(\frac{\gamma}{\delta}-\nu)\Gamma(\frac{\gamma}{\delta}-\nu)$$

$$\Gamma(\frac{\gamma}{\delta}-\nu) = \frac{\Gamma(\frac{\gamma}{\delta})}{(-1)^{\nu}(-\frac{\gamma}{\delta}+1)_{\nu}}$$

where, for example, $(a)_n = a(a+1)...(a+n-1), a \neq 0, (a)_0 = 1$ is the Pochhammer symbol. Also

$$\Gamma(-\frac{\gamma}{\rho}-\nu) = \frac{\Gamma(-\frac{\gamma}{\rho})}{(-1)^{\nu}(\frac{\gamma}{\rho}+1)_{\nu}}.$$

Then for $\rho = \delta$ and $\frac{\gamma}{\delta}$ not a positive integer, we have from (2.8)

$$\begin{aligned} B &= \frac{\Gamma(\frac{\gamma}{\delta})}{\rho a^{\frac{\gamma}{\rho}}}{}_0F_1(\ ;-\frac{\gamma}{\rho}+1;ab) \\ &+ \frac{\Gamma(-\frac{\gamma}{\delta})}{\rho}b^{\frac{\gamma}{\rho}}{}_0F_1(\ ;\frac{\gamma}{\rho}+1;ab). \end{aligned} \qquad (2.9)$$

Thus, it is the sum of two Bessel series. Hence Bessel integral is an appropriate name to be used for (1.1).

Special case (2): $\rho = \delta, \frac{\gamma}{\delta} = m, m = 1, 2, ...$

In this case the poles at $s = 0, -1, -2, ..., -(m-1)$ are simple and the poles at $s = -m, -m-1, ...$ are of order two each. In this case

$$B = \frac{1}{\delta a^{\frac{\gamma}{\delta}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(m+s)(ab)^{-s} \mathrm{d}s. \tag{2.10}$$

Sum of the residues at the poles $s = 0, -1, .., -(m-1)$ is given by

$$\frac{1}{\delta a^{\frac{\gamma}{\delta}}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \Gamma(m-\nu)(ab)^\nu. \tag{iii}$$

For $x = -m-\nu, \nu = 0, 1, ...$ the poles are of order two each. Let

$$\phi(s) = \Gamma(s)\Gamma(m+s)(ab)^{-s}.$$

Then the residue at the poles of order two, denoted by R_ν, is given by the following:

$$\begin{aligned}
R_\nu &= \lim_{s\to-\nu} \frac{\mathrm{d}}{\mathrm{d}s} \{(s+\nu)^2 \Gamma(s)\Gamma(m+s)(ab)^{-s}\} \\
&= \lim_{s\to-\nu} \frac{\mathrm{d}}{\mathrm{d}s} \{(s+\nu)^2 \frac{(s+\nu-1)^2...(s+m)^2(s+m-1)...s}{(s+\nu-1)^2...(s+m)^2(s+m-1)...s} \Gamma(s)\Gamma(m+s)(ab)^{-s}\} \\
&= \lim_{s\to-\nu} \frac{\mathrm{d}}{\mathrm{d}s} \{ \frac{\Gamma^2(s+\nu+1)}{(s+\nu-1)^2...(s+m)^2(s+m-1)...s} (ab)^{-s}\}
\end{aligned}$$

Note that $(ab)^{-s} = \mathrm{e}^{-s\ln(ab)}$ and

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi(s) = \phi(s) \frac{d}{\mathrm{d}s} \ln \phi(s).$$

Also

$$\begin{aligned}
\lim_{s\to-\nu} \phi(s) &= \lim_{s\to-\nu} \frac{\Gamma^2(s+\nu+1)}{(s+\nu-1)^2...(s+m)^2(s+m-1)...s} (ab)^{-s} \\
&= \frac{(-1)^m}{\nu!(\nu-m)!}(ab)^\nu, \nu = m, m+1, ...
\end{aligned}$$

$$\begin{aligned}
\lim_{s\to-\nu} \ln \phi(s) &= \lim_{s\to-\nu} \frac{\mathrm{d}}{\mathrm{d}s} \ln \phi(s) = \lim_{s\to-\nu} [2\psi(s+\nu+1) - \frac{2}{s+\nu-1} - ... - \frac{2}{s+m}] \\
&- \frac{1}{s+m-1} - ... - \frac{1}{s} - \ln(ab) \\
&= 2\psi(1) + 2[1 + \frac{1}{2} + ... + \frac{1}{\nu-m}] + (\frac{1}{\nu-m+1} + ... + \frac{1}{\nu}) - \ln(ab) \\
&= \psi(\nu+1) + \psi(\nu-m+1) - \ln(ab)
\end{aligned}$$

where $\psi(z)=\frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z)$ is the psi function. The above simplification is done by using the properties of the psi function. Hence

$$R_\nu=[\psi(\nu+1)+\psi(\nu-m+1)-\ln(ab)]\frac{(-1)^m}{\nu!(\nu-m)!}(ab)^\nu, \nu=m,m+1,... \quad (2.11)$$

Therefore

$$\begin{aligned}B=&\frac{1}{\delta a^{\frac{\gamma}{\delta}}}\{\sum_{\nu=0}^{m-1}\frac{(-1)^\nu}{\nu!}\Gamma(m-\nu)(ab)^\nu\\&+\sum_{\nu=m}^{\infty}[\psi(\nu+1)+\psi(\nu-m+1)-\ln(ab)][\frac{(-1)^m}{\nu!(\nu-m)!}(ab)^\nu]\}.\end{aligned}\quad (2.12)$$

By using the same procedure one can write the logarithmic version corresponding to (2.5) when the poles of $\Gamma(s)\Gamma(\frac{\gamma}{\delta}+\frac{\rho}{\delta}s)$ differ by integers. Since the expressions become too lengthy they are not listed here.

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