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THE CHARACTERISTIC RELATIONS DUE TO LUPAS- KUMAR-PATHAN- TYPE OPERATORS

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Dedicated to Prof. A.M. Mathai on his 80^{th} ***birth anniversary***

Abstract: In this work, we introduce Lupas-Kumar-Pathan -type operators and then study its convergence properties by using Cauchy-Schwarz inequalities of integration and summation and Chebyshev inequality of integration. We obtain the recurrence relations and some properties of these operators. These results are then applied with a view to obtaining some characteristic relations on central moments.

Keywords: Lupas operators, Kumar and Pathan type operators, characteristic results on central moments.

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1. Introduction

Lupas proposed a family of linear positive operators mapping $C[0,\infty)$ into $C[0,\infty)$, the class of all bounded and continuous functions on $C[0,\infty)$, namely (see Derriennic [1])

$$V_n(f,x) = \sum_{k=0}^{\infty} P_{n.k}(x) f\left(\frac{k}{n}\right),\ \forall x \in C[0,\infty),\ P_{n.k}(x) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{x^k}{(1+x)^{n+k}}. \tag{1}$$

Later on Sahai and Prasad [5] proposed a modification of Lupas type operators defined for functions integrable on $C[0,\infty)$ in the form

$$B_n(f,x) = (n-1)\sum_{k=0}^{\infty} P_{n.k}(x)\int_0^{\infty} P_{n.k}(t) f(t)dt, \tag{2}$$

$P_{n.k}(x)$ is given in equation (1).
Recently, Kumar and Pathan [3] defined a transformation formula for a bounded uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$H_k f(x) = \frac{1}{k}\int_{-\infty}^{\infty} f(u) g\left(\frac{u-x}{k}\right) du, \tag{3}$$

where $k > 0$, and $g(x)$ be a probability density function defined by $\int_{-\infty}^{\infty} g(x)dx = 1$, otherwise $g(x) = 0$.
Here, in this work we have defined a family of linear positive operators for the functions integrable on $[0, \infty)$ in the form

$$H^{\alpha,\beta}\{f(y)\}(x) = \sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x) \int_0^{\infty} G_n^{\alpha,\beta}(x, y) f(y) dy, \tag{4}$$

where $C_n^{\alpha,\beta}(x) = \begin{pmatrix} \alpha + (\beta + 1)n \\ n \end{pmatrix} \frac{x^n}{(1 + x)^{\beta n+n}}$ and

$$G_n^{\alpha,\beta}(x, y) = (\alpha + \beta n + 1)(1 + x)^n (1 + y)^{-\alpha-\beta n-2} \times$$

$$_2F_1\left[-n, \alpha + \beta n + 2; 1; \frac{xy}{(1 + x)(1 + y)}\right]$$

defined $\forall y \in [0, \infty)$, and otherwise, $G_n^{\alpha,\beta}(x, y) = 0$ and $\int_0^{\infty} G_n^{\alpha,\beta}(x, y)dy = 1$, provided that $\alpha \geq 0, 0 \leq \left|\frac{x}{(1 + x)^{\beta+1}}\right| < \left|\frac{\beta^\beta}{(\beta + 1)^{\beta+1}}\right|, \forall x \geq 0$. Here, the hypergeometric polynomial of degree n is given by

$$_2F_1[-n, a; c; x] = \sum_{r=0}^{n} \frac{(-n)_r (a)_r}{(c)_r} x^r \tag{5}$$

and the Pochhammer symbol is

$$(a)_r = a(a + 1)(a + 2)...(a + r - 1) \text{ and } (a)_0 = 1 \text{ (see Rainville [4])}$$

Put $f(x) = 1$ in Eqn. (4), and then use the result of Srivastava and Manocha [6, p. 355, Eqn. (5)] and Riordan array proofs of identities in Goulds book [Renzo

Sprugnoli, Dipartimento di Sistemi e Informatica, Viale Morgagni, 65 Firenze (Italy), February, 2006], under the conditions given in Eqns. (5), it reduces to

$$H^{\alpha,\beta}\{1\}(x) = \sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x) \int_0^{\infty} G_n^{\alpha,\beta}(x,y)dy = K^{\alpha,\beta}(x)$$

$$\Rightarrow \sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x) = K^{\alpha,\beta}(x) = \frac{(1+u(x))^{\alpha+1}}{(1-\beta u(x))},$$

where $\quad \dfrac{x}{(1+x)^{\beta+1}} = \dfrac{u(x)}{(1+u(x))^{\beta+1}}, u(0) = 0$ and $\left|\dfrac{x}{(1+x)^{\beta+1}}\right| < \left|\dfrac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\right|.$ (6)

Again, $K^{\alpha,\beta}(x) > 0,\ \forall\alpha \geq 0,\ \left|\dfrac{x}{(1+x)^{\beta+1}}\right| < \left|\dfrac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\right|$ and $x \geq 0$ Hence, the operator $H^{\alpha,\beta}\{f(y)\}(x)$ is a family of linear positive operators for the functions integrable on $[0,\infty)$ (see, Yuankwei and Shunsheng [8]). (7)

Further, put $\alpha = N-1$ and $\beta = 0$ in Eqn. (4) and use Eqns. (1), (2), (5) and (6) to get relation with Derriennic [1] and Sahai and Prasad [5] operators

$$H^{N-1,0}\{1\}(x) = \sum_{n=0}^{\infty} C_n^{N-1,0}(x) = B_N(1,x) = \frac{1}{(1+x)^{N-1}} \tag{8}$$

Hence due to Eqns. (7) and (8), with the conditions given in Eqn. (5), the operator given in Eqn. (4), may be defined by Lupas-Kumar-Pathan type operators.

In this work, we study the convergence properties and applications to obtain some characteristic results on approximations.

2. The Convergence Properties of the Operator $H^{\alpha,\beta}\{f(y)\}(x)$

In this section, on using Cauchy Schwarz inequalities of integration and summation (see Steele [7]) and Chebyshev inequality of integration , defined in the form (see Devore and Berk [2] and Steele [7])

Let Y be a real continuous random variables with the mean $\mu = \int_{-\infty}^{\infty} yg(y)dy$ and the variance $V(y) = \sigma^2 = \int_{-\infty}^{\infty}(y-\mu)^2 g(y)dy$ here, $g(y)$ is a probability density function defined $\forall y \in (-\infty,\infty)$.

Then, $\int_{-\infty}^{\mu-\delta}(y-\mu)^2 g(y)dy + \int_{\mu+\delta}^{\infty}(y-\mu)^2 g(y)dy \leq \sigma^2$, and again, $P(|Y-\mu| \geq \delta) \leq \frac{\sigma^2}{\delta^2}$ which implies that

$$\int_{-\infty}^{\mu-\delta} g(y)dy + \int_{\mu+\delta}^{\infty} g(y)dy \leq \frac{\sigma^2}{\delta^2}. \tag{9}$$

we obtain the convergence properties of the operator $H^{\alpha,\beta}\{f(y)\}(x)$ given in Eqn. (4). The mean μ of given distribution is defined in Eqn. (9).
Consider the inequality

$$|H^{\alpha,\beta}\{f(y)\}(x)-f(\mu)| \leq |H^{\alpha,\beta}\{f(y)\}(x)-H^{\alpha,\beta}\{f(\mu)\}(x)|+|H^{\alpha,\beta}\{f(\mu)\}(x)-f(\mu)|$$

$$\leq \left|\sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x)\int_0^{\infty} G_n^{\alpha,\beta}(x,y)\{f(y)-f(\mu)\}dy\right| + |f(\mu)\{K^{\alpha,\beta}(x)-1\}| \tag{10}$$

Now use Cauchy-Schwarz inequality of integration in inequality (10) to get that

$$|H^{\alpha,\beta}\{f(y)\}(x)-f(\mu)|$$
$$\leq \left|\sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x)\left(\int_0^{\infty} G_n^{\alpha,\beta}(x,y)dy\right)^{\frac{1}{2}}\left(\int_0^{\infty} G_n^{\alpha,\beta}(x,y)\{f(y)-f(\mu)\}^2dy\right)^{\frac{1}{2}}\right|$$
$$+|f(\mu)||K^{\alpha,\beta}(x)-1| \tag{11}$$

But $\int_0^{\infty} G_n^{\alpha,\beta}(x,y)dy = 1$ (see Eqn. (6)), so that on using Cauchy-Schwarz inequality of summation in inequality (11), it becomes

$$|H^{\alpha,\beta}\{f(y)\}(x)-f(\mu)|$$
$$\leq \left|\left(\sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x)\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x)\int_0^{\infty} G_n^{\alpha,\beta}(x,y)\{f(y)-f(\mu)\}^2dy\right)^{\frac{1}{2}}\right|$$
$$+|f(\mu)||K^{\alpha,\beta}(x)-1| \tag{12}$$

Now, let $|f(y)-f(\mu)| < \varepsilon$ for $|y-\mu| < \delta$, then the inequality (12) may be written as

$$|H^{\alpha,\beta}\{f(y)\}(x)-f(\mu)|$$
$$\leq \left|\left(\sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x)\right)^{\frac{1}{2}}\left(\frac{\varepsilon^2}{\delta^2}\sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x)\int_0^{\infty} |y-\mu|^2 G_n^{\alpha,\beta}(x,y)dy\right)^{\frac{1}{2}}\right|$$
$$+|f(\mu)||K^{\alpha,\beta}(x)-1|\left\{\frac{1}{\delta^2}\int_0^{\infty} |y-\mu|^2 G_n^{\alpha,\beta}(x,y)dy\right\}^{\frac{1}{2}} \tag{13}$$

Again, making an application of the inequality (9) and Eqn. (6) in Eqn. (13), (as $K^{\alpha,\beta}(x) > 0 \ \ \forall \alpha \geq 0$, $\left|\frac{x}{(1+x)^{\beta+1}}\right| < \left|\frac{\beta^\beta}{(\beta+1)^{\beta+1}}\right|$ and $x \geq 0$, we get

$$|H^{\alpha,\beta}\{f(y)\}(x)-f(\mu)| \leq \frac{\varepsilon\sigma}{\delta^2}|K^{\alpha,\beta}(x)| + |f(\mu)||\{K^{\alpha,\beta}(x)-1\}|\frac{\sigma}{\delta^2}$$

$$\leq \frac{\varepsilon\sigma}{\delta^2}|K^{\alpha,\beta}(x)| + |K^{\alpha,\beta}(x)+1|\frac{M\sigma}{\delta^2} \leq \frac{\varepsilon\sigma}{\delta^2} + 2\frac{M\sigma}{\delta^2} \quad (14)$$

Now ε is arbitrary and then set $\varepsilon \to 0$, $|f(y)| < M$ and choose $\delta >> M\sigma$ in Eqn. (14), $H^{\alpha,\beta}\{f(y)\}(x)$ converges to $f(\mu)\forall y \in [0,\infty)$ and $x \geq 0$. We may say that under the conditions of Eqn. (5), $\varepsilon \to 0$, $|f(y)| < M$ and $\delta >> M\sigma$, there exists

$$H^{\alpha,\beta}\{f(y)\}(x) \to f(\mu), \quad \forall\ y \in\ [0,\infty) \ \text{ and } \ x \geq 0. \quad (15)$$

3. The Characteristic Relations of Operator $H^{\alpha,\beta}\{f(y)\}(x)$ with Central Moments due to density $G_n^{\alpha,\beta}(x,y)$ Auxiliary Relations

On using the formula of mean given in Eqn. (9) due to density $G_n^{\alpha,\beta}(x,y)$ defined in Eqn. (5), we have

$$\mu(x) = \frac{(1+x)^n}{(\alpha+\beta n)}\ {}_2F_1\left[-n,2;1;\frac{x}{1+x}\right] = \frac{1}{(\alpha+\beta n)}[1-nx],$$

such that

$$\mu(0) = \frac{1}{(\alpha+\beta n)} \text{ and } \mu(1) = \frac{(2)^n}{(\alpha+\beta n)}\ {}_2F_1\left[-n,2;1;\frac{1}{2}\right], \text{ for all } n = 0,1,2,... \quad (16)$$

Again, the $m-th$ central moment due to density $G_n^{\alpha,\beta}(x,y)$ is defined by

$$M_{m,n}^{\alpha,\beta}(x;\mu(x)) = \int_0^\infty (y-\mu(x))^m G_n^{\alpha,\beta}(x,y)dy \quad (17)$$

Therefore, making use of Eqns. (4) and (5) in Eqn. (17), we find the relation of the operator $H^{\alpha,\beta}\{f(y)\}(x)$ with central moment due to density $G_n^{\alpha,\beta}(x,y)$ in the form

$$H^{\alpha,\beta}\{(y-\mu(x))^m\}(x) = \sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x) M_{m,n}^{\alpha,\beta}(x;\mu(x)) \quad (18)$$

Further, from Eqn. (17), we may write

$$M_{m,n}^{\alpha,\beta}(x;\mu(x)) = \sum_{s=0}^{m} \left(\begin{array}{c} m \\ s \end{array}\right) \left(\begin{array}{c} \alpha+\beta n \\ s \end{array}\right)^{-1} (1+x)^n$$

$$\times\ {}_2F_1\left[-n,s+1;1;\frac{x}{1+x}\right](-\mu(x))^{m-s} \quad (19)$$

Here, for all $n = 0,1,2,...$, and $x \geq 0$, $\left(\begin{array}{c} m \\ s \end{array}\right)^{-1} = \frac{s!(\alpha-s)!}{\alpha!}$, the hypergeometric polynomial of degree n is given by Eqn. (5).

Now, put $m = 1, 2, 3, 4$ in Eqn. (19) and use the result (16), we may compute the first, second, third and fourth moments respectively by following formula

$$M_{1,n}^{\alpha,\beta}(x;\mu(x)) = \sum_{s=0}^{1} \begin{pmatrix} 1 \\ s \end{pmatrix} \begin{pmatrix} \alpha+\beta n \\ s \end{pmatrix}^{-1} (1+x)^n$$

$$\times\ {}_2F_1\left[-n, s+1; 1; \frac{x}{1+x}\right](-\mu(x))^{1-s} = 0,\ \text{(always)},$$

$$M_{2,n}^{\alpha,\beta}(x;\mu(x)) = \sum_{s=0}^{2} \begin{pmatrix} 2 \\ s \end{pmatrix} \begin{pmatrix} \alpha+\beta n \\ s \end{pmatrix}^{-1} (1+x)^n$$

$$\times {}_2F_1\left[-n, s+1; 1; \frac{x}{1+x}\right](-\mu(x))^{2-s}$$

$$= \left[(\mu(x))^2 + 2\mu(x)\frac{(nx-1)}{(\alpha+\beta n)} + \frac{(n(n-1)x^2 - 4nx + 2)}{(\alpha+\beta n)(\alpha+\beta n-1)}\right],$$

$$M_{3,n}^{\alpha,\beta}(x;\mu(x)) = \sum_{s=0}^{3} \begin{pmatrix} 3 \\ s \end{pmatrix} \begin{pmatrix} \alpha+\beta n \\ s \end{pmatrix}^{-1} (1+x)^n$$

$$\times {}_2F_1\left[-n, s+1; 1; \frac{x}{1+x}\right](-\mu(x))^{3-s}$$

$$= \left[-(\mu(x))^3 - 2(\mu(x))^2\frac{(nx-1)}{(\alpha+\beta n)} - 3\mu(x)\frac{(n(n-1)x^2 - 4nx + 2)}{(\alpha+\beta n)(\alpha+\beta n-1)}\right.$$

$$\left. + \frac{(6 - 18nx + 9n(n-1)x^2 - n(n-1)(n-2)x^3)}{(\alpha+\beta n)(\alpha+\beta n-1)(\alpha+\beta n-2)}\right],$$

and $M_{4,n}^{\alpha,\beta}(x;\mu(x)) = \sum_{s=0}^{4} \begin{pmatrix} 4 \\ s \end{pmatrix} \begin{pmatrix} \alpha+\beta n \\ s \end{pmatrix}^{-1} (1+x)^n$

$$\times {}_2F_1\left[-n, s+1; 1; \frac{x}{1+x}\right](-\mu(x))^{4-s}$$

$$= \left[(\mu(x))^4 + 4(\mu(x))^3\frac{(nx-1)}{(\alpha+\beta n)} + 6(\mu(x))^2\frac{(n(n-1)x^2 - 4nx + 2)}{(\alpha+\beta n)(\alpha+\beta n-1)}\right.$$

$$+ 4\mu(x)\frac{(n(n-1)(n-2)x^3 - 9n(n-1)x^2 + 18nx - 6)}{(\alpha+\beta n)(\alpha+\beta n-1)(\alpha+\beta n-2)}$$

$$+\frac{(n(n-1)(n-2)(n-3)x^4-16n(n-1)(n-2)x^3+72n(n-1)x^2-96nx+24)}{(\alpha+\beta n)(\alpha+\beta n-1)(\alpha+\beta n-2)(\alpha+\beta n-3)}\Bigg] \quad (20)$$

4. Main relations

Theorem 1 If $a, b, c \in \mathbb{R}$ (the set of real numbers) such that $b^2 \geq 4ac$, $a > 0$ and set $(y-\mu(x)) \geq \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$, $\forall x \geq 0$, then following inequalities hold

(i) $M_{4,n}^{\alpha,\beta}(x;\mu(x)) \geq \left(\frac{c}{a}\right)^2$

(ii) $a^2 M_{4,n}^{\alpha,\beta}(x;\mu(x)) + 2ab M_{3,n}^{\alpha,\beta}(x;\mu(x)) + (b^2+2ac) M_{2,n}^{\alpha,\beta}(x;\mu(x)) + c^2 \geq 0.$

(iii)

$$\begin{vmatrix} M_{4,n}^{\alpha,\beta}(x;\mu(x)) & M_{3,n}^{\alpha,\beta}(x;\mu(x)) & M_{2,n}^{\alpha,\beta}(x;\mu(x)) \\ M_{3,n}^{\alpha,\beta}(x;\mu(x)) & M_{2,n}^{\alpha,\beta}(x;\mu(x)) & 0 \\ M_{2,n}^{\alpha,\beta}(x;\mu(x)) & 0 & 1 \end{vmatrix} \geq 0. \quad (21)$$

Here, for the consistency conditions, it should be $0 \leq M_{2,n}^{\alpha,\beta}(x;\mu(x)) < \infty$, $M_{4,n}^{\alpha,\beta}$ $(x;\mu(x)) > (M_{2,n}^{\alpha,\beta}(x;\mu(x)))^2$ and $M_{4,n}^{\alpha,\beta}(x;\mu(x)) M_{2,n}^{\alpha,\beta}(x;\mu(x)) > (M_{3,n}^{\alpha,\beta}(x;\mu(x)))^2$.

Proof: **(i)** Since $(y-\mu(x)) \geq \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$, $\forall x \geq 0, a > 0$, thus $a(y-\mu(x))^2 + b(y-\mu(x)) + c \geq 0$.
Therefore,

$$\int_0^\infty \left\{a(y-\mu(x))^2 + b(y-\mu(x)) + c\right\} G_n^{\alpha,\beta}(x,y)dy \geq 0. \quad (22)$$

Now, use Eqns. (16), (17) and (20) in inequality (22) and then apply Chauchy-Schwarz inequality (Steele [7]) we get,

$$0 \leq \int_0^\infty \left\{a(y-\mu(x))^2 + b(y-\mu(x)) + c\right\} G_n^{\alpha,\beta}(x,y)dy \leq \left\{\int_0^\infty G_n^{\alpha,\beta}(x,y)dy\right\}^{1/2}$$

$$\left\{\int_0^\infty \left\{a(y-\mu(x))^2 + b(y-\mu(x)) + c\right\}^2 G_n^{\alpha,\beta}(x,y)dy\right\}^{1/2} \quad (23)$$

and then using above consistency condition and Eqn. (22), we get the result (21, (i)).

(ii) Again, use Eqns. (5),(16), (17) and (20) in inequality (23), we get the result

(21, (ii)).
(iii) Further, the result (21,(ii)) is second order homogeneous equation in (a,b,c) hence applying one of the properties of second order homogeneous equation in it, we find the result (21,(iii)). On solving above determinant given in Eqn. (21,(iii)), we also find above consistency conditions.

Theorem 2. If the given data are consistence, then it is followed that $M_{4,n}^{\alpha,\beta}(x;\mu(x)) > (M_{2,n}^{\alpha,\beta}(x;\mu(x)))^2$ and $M_{4,n}^{\alpha,\beta}(x;\mu(x))M_{2,n}^{\alpha,\beta}(x;\mu(x)) > (M_{3,n}^{\alpha,\beta}(x;\mu(x)))^2$ and thus

$$\sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x)(M_{2,n}^{\alpha,\beta}(x;\mu(x)))^2 < H^{\alpha,\beta}\{(y-\mu(x))^4\}(x) \tag{24}$$

$$\sum_{n=0}^{\infty} C_n^{\alpha,\beta}(x)(M_{3,n}^{\alpha,\beta}(x;\mu(x)))^2 < [H^{\alpha,\beta}\{(y-\mu(x))^2\}(x)]^{1/2}[H^{\alpha,\beta}\{(y-\mu(x))^4\}(x)]^{1/2}. \tag{25}$$

Proof. Use the result $(M_{2,n}^{\alpha,\beta}(x;\mu(x)))^2 < M_{4,n}^{\alpha,\beta}(x;\mu(x))$ and multiply both of its sides by $C_n^{\alpha,\beta}(x)$ and then sum them from $n = 0$ to $n = \infty$, and again using Chauchy-Schwarz inequality of summation, we get the result (24).

Further, use the result $(M_{3,n}^{\alpha,\beta}(x;\mu(x)))^2 < M_{4,n}^{\alpha,\beta}(x;\mu(x))M_{2,n}^{\alpha,\beta}(x;\mu(x))$, and apply same techniques to get the result (25).

5. Applications

In this section, we apply above results to obtain following approximations,

Let α, β be given in Eqn. (5) and a and c be arbitrary given by Eqn. (21). Then applying Eqns. (16), (20) and (22) .we get the inequalities

$$ {}_2F_1\left[-n,3;1;\frac{x}{1+x}\right] + \left(\frac{1}{(\alpha+\beta n)}-1\right)\left\{{}_2F_1\left[-n,2;1;\frac{x}{1+x}\right]\right\}^2 \geq -\frac{c}{a}(\alpha+\beta n-1)(\alpha+\beta n)(1+x)^n \tag{26}$$

Again, let α, β be given in Eqn. (5), a and c be arbitrary given by Eqn. (21) and λ_n be any bounded sequence. Then form Eqn. (26), for $|t| < 1$, we find the inequality

$$\sum_{n=0}^{\infty}\lambda_n\, {}_2F_1\left[-n,3;1;\frac{x}{1+x}\right]t^n + \sum_{n=0}^{\infty}\lambda_n\left(\frac{1}{(\alpha+\beta n)}-1\right) \left\{{}_2F_1\left[-n,2;1;\frac{x}{1+x}\right]\right\}^2 t^n + \frac{c}{a}\sum_{n=0}^{\infty}\lambda_n(\alpha+\beta n-1)(\alpha+\beta n)\{(1+x)t\}^n \geq 0. \tag{27}$$

By theory of generating functions and application of the Eqns. (26) and (27), we can obtain various inequalities for known and less known summation formulas involving various polynomials.

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