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SPHERICALLY SYMMETRIC ANISOTROPIC PERFECT FLUID BALL IN GENERAL RELATIVITY

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Dedicated to Prof. A.M. Mathai on his 80^{th} birth anniversary

Abstract: In this investigation of a spherically symmetric shear free anisotropic fluid we present a new model of the general relativistic field equations by using Tewari and Charan [1] solution as a seed solution. The solution is having positive finite central pressures and central density. The ratio of pressures and density is less than one and casualty condition is obeyed at the centre. Further, the outmarch of pressures, density and pressure-density ratio, and the ratio of sound speed to light is monotonically decreasing. The central red shift and surface red shift are positive and monotonically decreasing. Further by assuming the suitable surface density, we have constructed a compact star model with all degree of suitability.

Keywords: Exact solutions, Einsteins field equations, Perfect fluid ball, Compact star, General relativity.

1. Introduction

A compact stellar object is formed by an equilibrium state which is reached after condensation and contraction of a massive gas cloud. At this state thermal radiation pressure together with normal fluid pressure balances the gravitational binding energy. Various studies are made for understanding the formation of compact star, its physical properties and internal structure by the solution of Einstein's field equation. Therefore the static isotropic and anisotropic exact solution which describes the compact star is caused to enthusiasts the Researchers to conduct the work in the same field. The study of interior of massive fluid ball can be made by well behaved solution of Einstein's field equation. These equations were solved by Schwarzschild for the interior of the static compact stellar object. The first ever two exact solution of Einstein field equation for a compact object in static equilibrium was obtained by Schwarzschild [2] in 1916. The first solution corresponds to the geometry of the space-time exterior to a static prefect fluid ball, while the other solution describes the interior geometry of a fluid sphere of constant energy-density. Tolman [3] has obtained five different types of exact solutions for static cases. The III solution corresponds to the constant density solution obtained earlier by Schwarzschild [2]. The V and VI solutions correspond to infinite density and infinite pressure at the centre, hence not considered physically viable. Thus only the IV and VII solutions of Tolman are of physical relevance. Despite the non linear character of Einsteins field equations, various exact solutions for static and spherically symmetric metric are available in the related literature.

The search for the exact solutions is of continuous interest to researcher. Buchdahl [4] proposed a famous bound on the mass radius ratio of relativistic fluid spheres which is an important contribution in order to study the stability of the fluid spheres. Delgaty-Lake [5] studied all the then existing solutions and established that Adler [6], Heintzmann [7], Finch and Skea [8], etc. do not satisfy all the well behaved conditions and also pointed out that only nine solutions are well behaved; out of which seven in curvature coordinates (Tolman [3], Patvardhav and Vaidya [9], Mehra [10], Kuchowicz [11], Matese and Whitman [12], Durgapals two solutions [13] and only two solutions (Nariai [14], Goldman [15]) in isotropic coordinates. Ivanov [16], Neeraj Pant [17], Maurya and Gupta [18], Pant et al. ([19],[20]) studied the existing well behaved solutions of Einstein field equations in isotropic coordinates. Recently we have found some exact solutions of Einsteins field equations given by Tewari [21], Tewari and Charan ([22]-[25]).

In this paper we present a new solution in spherically symmetric isotropic coordinates which is well behaved. Keeping in view of generality of solution due to Tewari and Charan [1], we present a special solution of the same and its detailed study, in order to construct a realistic model of compact star. In our present study the paper consists of seven sections. In section 2 Einstein's field equations in isotropic coordinates are given. Expressions of density, anisotropic pressures(radial and transverse pressures), anisotropy constant and redshift are incorporated in this section. Section 3 consists of boundary conditions for well behaved solutions. A new class of solution of Einstein's field equations in isotropic coordinates is given in section 4. Section 5 stipulates the properties of this new class of solution of Einstein's field equations. In section 6 the matching conditions of interior metric of the perfect fluid with the Schwarzschild exterior metric are given. Finally, some concluding remarks have been made in section 7.

2. Einstein's Field Equation in anisotropic coordinates

The Einstein's field equations of general relativity are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

where $T_{\mu\nu}$, the energy momentum tensor for a perfect fluid ball is defined as

$$T_{\mu\nu} = (\rho c^2 + p_r)u_\mu u_\nu - p_t g_{\mu\nu} + (p_r - p_t)x_\mu x_\nu \tag{2}$$

where ρ is the proper density, p_r and p_t are pressures of the fluid in the direction of u_μ(radial pressure)and orthogonal to u_μ (tangential pressure) respectively, u_μ time-like four-velocity vector, x_μ is the unit space like vector in the direction of radial vector and g^μ_ν metric tensor of space-time.

The interior space-time metric for spherically symmetric fluid distribution is given by

$$ds^2 = -B^2\{dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\} + A^2dt^2 \tag{3}$$

where A and B are functions of r only.

In view of the metric (3) and energy momentum tensor (2), the field equation (1) gives

$$\frac{8\pi G}{c^4}p_r = \frac{1}{B^2}\Big(\frac{B'^2}{B^2} + \frac{2B'}{rB} + \frac{2A'B'}{AB} + \frac{2A'}{rA}\Big) \tag{4}$$

$$\frac{8\pi G}{c^4}p_t = \frac{1}{B^2}\Big(\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{B'}{rB} + \frac{A''}{A} + \frac{A'}{rA}\Big) \tag{5}$$

$$\frac{8\pi G}{c^2}\rho = -\frac{1}{B^2}\Big(\frac{2B''}{B} - \frac{B'^2}{B^2} + \frac{4B'}{rB}\Big) \tag{6}$$

$$\frac{8\pi G}{c^4}(p_t - p_r) = \Delta(r) = \frac{\delta(r)}{B^2} \tag{7}$$

where

$$\delta(r) = \Big(\frac{A''}{A} + \frac{B''}{B} - \frac{2B'^2}{B^2} - \frac{B'}{rB} - \frac{2A'B'}{AB} - \frac{A'}{rA}\Big) \tag{8}$$

The gravitational redshift of massive spherically symmetric ball is

$$1 + Z = g_{00}^{\frac{1}{2}} \tag{9}$$

which gives central (Z_0) and surface (Z_Σ) gravitational redshifts

$$Z_0 = \frac{c}{A} - 1 \tag{10}$$

and

$$Z_\Sigma = \left(1 + \frac{rB'}{B}\right)^{-1} - 1 \tag{11}$$

3. Boundary conditions for well behaved Solution

For well behaved nature of the solution in isotropic coordinates, the following conditions should be satisfied (Bonnor-Vickers [26]):

(i) The solution should be free from geometrical and physical singularities. Metric potentials A and B must be non-zero positive finite for free from geometrical singularities while central pressure, central density, should be positive and finite or $\rho_0 > 0$ and $p_0 > 0$ for free from physical singularities.

(ii) The solution should have maximum positive values of pressure and density at the center and monotonically decreasing towards the surface of fluid object i.e.

(a) $(\frac{dp_r}{dr})_0 = 0$ and $(\frac{d^2p_r}{dr^2})_0 < 0$ such that the radial pressure gradient, $\frac{dp_r}{dr}$ is negative for $0 \leq r \leq r_\Sigma$.

(b) $(\frac{dp_t}{dr})_0 = 0$ and $(\frac{d^2p_t}{dr^2})_0 < 0$ such that the tangential pressure gradient, $\frac{dp_t}{dr}$ is negative for $0 \leq r \leq r_\Sigma$.

(c) $(\frac{d\rho}{dr})_0 = 0$ and $(\frac{d^2\rho}{dr^2})_0 < 0$ such that the density gradient, $\frac{d\rho}{dr}$ is negative for $0 \leq r \leq r_\Sigma$.

(iii) The radial pressure must be equal to the tangential pressure at the center i.e. $(p_r)_0 = (p_t)_0$.

(iv) At boundary radial pressure, p_r must vanish while tangential pressure, p_t may not vanish.

(v) The radial pressure, p_r, tangential pressure, p_t and density ρ should be positive.

(vi) Solution should have positive value of pressure-density ratio which must be less than 1 (weak energy condition) and less than (strong energy condition) throughout within the fluid object and monotonically decreasing as well. (Pant and Negi [27]).

(vii) The casualty condition must be satisfied for this velocity of sound should be less than that of light throughout the model i.e.$0 \leq \sqrt{\frac{dp_r}{c^2 d\rho}} < 1$ and $0 \leq \sqrt{\frac{dp_t}{c^2 d\rho}} < 1$. The velocity of sound should be monotonically decreasing towards the surface and increasing with the increase of density i.e. $\frac{d}{dr}(\frac{dp_r}{d\rho}) < 0$ or $(\frac{d^2p_r}{d\rho^2}) > 0$ and

$\frac{d}{dr}(\frac{dp_t}{d\rho}) < 0$ or $(\frac{d^2p_t}{d\rho^2}) > 0$. In this context it is worth mentioning that the equation of state at ultra-high distribution has the property that the sound speed is decreasing outwards. (Canuto and Lodenquai [28]).

(viii) The anisotropy factor Δ should be zero at the center and increasing towards the surface of fluid object.

(ix) For realistic matter, $\gamma > 1$ i.e. $\frac{p}{\rho} < \frac{dp}{d\rho}$, everywhere within the ball. (Pant and Maurya [29])

(x) The red shift at the center z_0 and at the boundary should be positive, finite and monotonically decreasing in nature with the increase of r.

Under these conditions, we have to assume the one of the gravitational potential component in such a way that the field equation (1) can be integrated and solution should be well behaved.

4. New class of well behaved Solution

A class of solutions of (8) is obtained by Tewari and Charan [1] as follows

$$A_0 = C_4(1 + C_3r^2)(1 + C_1r^2)^{\frac{n}{l+1}} \tag{12}$$

$$B_0 = C_2(1 + C_1r^2)^{\frac{1}{l+1}} \tag{13}$$

$$\delta(r) = \frac{(2n-2)}{(l+1)}\frac{4C_3C_1r^2}{(1+C_3r^2)(1+C_1r^2)} \tag{14}$$

where n, l, C_1, C_2, C_3 and C_4 are constants and

$$n = \frac{1}{2}\Big\{(l + 3) \pm (l^2 + 10l + 17)^{\frac{1}{2}}\Big\} \tag{15}$$

where n is real if $l \geq -5 + 2\sqrt{2}$ or $l \leq -5 - 2\sqrt{2}$.

For different values of n or l above equations give a variety of solutions and they are categorized as isotropic pressure and homogeneous density, isotropic pressure and inhomogeneous density while some inhomogeneous density and anisotropic pressure. In order to keep anisotropy in our mind we are here constructing a specific model corresponding to $n = -\frac{7}{5}$, and we have

$$A = C_4(1 + C_3r^2)(1 + C_1r^2)^{\frac{7}{47}} \tag{16}$$

$$B = C_2(1 + C_1r^2)^{\frac{-5}{47}} \tag{17}$$

$$\delta(r) = \frac{96C_3C_1r^2}{47(1+C_1r^2)(1+C_3r^2)} \tag{18}$$

The energy density, anisotropic pressures and anisotropic parameter of star are given by

$$\frac{8\pi G}{c^4}p_r = \frac{4C_1}{2209C_2^2(1+C_1r^2)^{\frac{37}{47}}}[94+49C_1r^2+\frac{47C_3(1+C_1r^2)(47+37C_1r^2)}{C_1(1+C_3r^2)}] \quad (19)$$

$$\frac{8\pi G}{c^4}p_t = \frac{4C_1}{2209C_2^2(1+C_1r^2)^{\frac{84}{47}}}[94+49C_1r^2+\frac{47C_3(1+C_1r^2)(47+61C_1r^2)}{C_1(1+C_3r^2)}] \quad (20)$$

$$\frac{8\pi G}{c^2}\rho = \frac{20C_1(141+42C_1r^2)}{2209C_2^2(1+C_1r^2)^{\frac{84}{47}}} \quad (21)$$

$$\Delta(r) = \frac{96C_3C_1r^2}{47C_2^2(1+C_1r^2)^{\frac{37}{47}}(1+C_3r^2)} \quad (22)$$

In view of equation (19) and (20) the rate of fall of pressures with radial distance from the center p_r' and p_t' are given by

$$\frac{8\pi G}{c^4}p_r' = \frac{8r}{103823C_2^2(1+C_1r^2)^{\frac{37}{47}}}\Big[\frac{-1175C_1^2+490C_1^3r^2}{(1+C_1r^2)}$$

$$+\frac{470C_1C_3-2209C_3^2+(1749C_1^2C_3-2099C_1C_3^2)r^2+370C_1^2C_3^2r^4}{(1+C_3r^2)^2}\Big] \quad (23)$$

$$\frac{8\pi G}{c^4}p_t' = \frac{4r}{103823C_2^2(1+C_1r^2)^{\frac{84}{47}}}\Big[\frac{-5593C_1^2-1813C_1^3r^2}{(1+C_1r^2)}$$

$$+\frac{35814C_1C_3-103823C_3^2+(11468C_1^2C_3-202758C_1C_3^2)r^2+123281C_1^2C_3^2r^4}{(1+C_3r^2)^2}\Big] \quad (24)$$

In view of equation (21) the rate of fall of density with radial distance from the center ρ' is given by

$$\frac{8\pi G}{c^2}\rho' = -\frac{1680C_1^2r(235+37C_1r^2)}{103823C_2^2(1+C_1r^2)^{\frac{131}{47}}} \quad (25)$$

5. Properties of new solution

The solution should be free from singularities i.e. central pressure, central density, should be positive and finite. For this A and B must be positive or $C_4 \geq 0$ and $C_2 \geq 0$.

The central pressure and density of star are given by

$$\frac{8\pi G}{c^4}(p_r)_0 = \frac{4(2209C_3 + 94C_1)}{2209C_2^2} \tag{26}$$

$$\frac{8\pi G}{c^4}(p_t)_0 = \frac{4(2209C_3 + 94C_1)}{2209C_2^2} \tag{27}$$

$$\frac{8\pi G}{c^2}\rho_0 = \frac{60C_1}{47C_2^2} \tag{28}$$

The central value of pressure and density is positive definite if $2209C_3 + 94C_1 > 0$ and $C_1 > 0$. For the values of C_1 and C_3 such that $\frac{2209C_3 + 94C_1}{705C_1} \leq 1$ or $\frac{C_3}{C_1} \leq \frac{656}{2209}$; the central value $\frac{p_0}{C^2\rho_0} \leq 1$. At the center,

$$(p_r')_0 = 0; \qquad (p_t')_0 = 0 \qquad \textit{and} \qquad (\rho')_0 = 0 \tag{29}$$

$$\frac{8\pi G}{c^4}(p_r'')_0 = -\frac{-8(25C_1^2 - 10C_1C_3 + 43C_3^2)}{2209C_2^2} \tag{30}$$

and

$$\frac{8\pi G}{c^4}(p_t'')_0 = -\frac{-4(119C_1^2 - 762C_1C_3 + 2209C_3^2)}{2209C_2^2} \tag{31}$$

Equations (30) and (31) give negative value of $(p_r'')_0$ and $(p_t'')_0$ for all values of C_1, C_2 and C_3 which satisfy boundary conditions. Hence pressure is maximum at the centre and monotonically decreasing.

At the center the value of ρ'' is given by

$$\frac{8\pi G}{c^2}(\rho'')_0 = -\frac{8400C_1^2}{2209C_2^2} \tag{32}$$

which is always negative for all values of C_1 and C_2. Thus density is maximum at center and is monotonically decreasing.

Square of the components of adiabatic sound speed at the center is given by

$$\frac{1}{c^2}\left(\frac{dp_r}{d\rho}\right)_0 = \frac{(25C_1^2 - 10C_1C_3 + 47C_3^2)}{1050C_1^2} \tag{33}$$

and

$$\frac{1}{c^2}\left(\frac{dp_t}{d\rho}\right)_0 = \frac{(119C_1^2 - 762C_1C_3 + 2209C_3^2)}{2100C_1^2} \tag{34}$$

The casuality condition is obeyed at the center for all values of constant satisfying the boundary conditions.

Further it is mentioned here that the boundary of the super dense star is established only when $C_4 > 0$ and $C_2 > 0$, $\dfrac{-1-2\sqrt{482}}{205} < \dfrac{C_3}{C_1} < \dfrac{-1+2\sqrt{482}}{205}$.

6. Matching Conditions of Boundary

Schwarzschild exterior solution in canonical coordinates is given as so obtained are to be matched over the boundary with Schwarzschild exterior solution

$$ds^2 = -\Big(1 - \frac{2GM}{c^2 r}\Big)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \Big(1 - \frac{2GM}{c^2 r}\Big)c^2 dt^2 \tag{35}$$

where M is the mass of the ball determined by external observer and R is the radial coordinate of the exterior region. Using the following transformation

$$r = R\Big(1 + \frac{GM}{2c^2 R}\Big)^2 \tag{36}$$

Eq.(35) can be transformed in isotropic form given as

$$ds^2 = -\Big(1 + \frac{GM}{2c^2 R}\Big)^4 \Big\{dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)\Big\} + \frac{\Big(1 - \frac{GM}{2c^2 R}\Big)^2}{\Big(1 + \frac{GM}{2c^2 R}\Big)^2} c^2 dt^2 \tag{37}$$

The usual boundary conditions are that the first and second fundamental forms are continuous over the boundary $r = r_\Sigma$ or equivalently $R = R_\Sigma$. Therefore

$$C_4(1 + C_3 r_\Sigma^2)(1 + C_1 r_\Sigma^2)^{\frac{7}{47}} = c^2\Big(\frac{2 - S_p}{2 + S_p}\Big) \tag{38}$$

$$r_\Sigma = R_\Sigma C_2 (1 + C_1 r_\Sigma^2)^{\frac{-5}{47}} \tag{39}$$

$$4C_2(1 + C_1 r_\Sigma^2)^{\frac{-5}{47}} = (2 + S_p)^2 \tag{40}$$

$$\Big(\frac{B'}{B} + \frac{1}{r}\Big)_{r_\Sigma} r_\Sigma = (1 - 2S_p)^{\frac{1}{2}} \tag{41}$$

$$\Big(\frac{A'}{A}\Big)_{r_\Sigma} r_\Sigma = S_p(1 - 2S_p)^{\frac{-1}{2}} \tag{42}$$

where $S_p = \frac{GM}{c^2 R_\Sigma}$.

In view of the above boundary conditions we get the values of the arbitrary constants in terms of Schwarzschild parameter S_p .

$$C_1 = \frac{(1-2S_p)^{\frac{1}{2}} - 47}{r_\Sigma^2(37-(1-2S_p)^{\frac{1}{2}})} \tag{43}$$

$$C_2 = \frac{(2+S_p)^2}{4}\left\{\frac{10}{(1-2S_p)^{\frac{1}{2}} - 37}\right\}^{\frac{5}{47}} \tag{44}$$

$$C_3 = \frac{7C_1 - \frac{47S_p(1-2S_p)^{\frac{-1}{2}}(1+C_1r_\Sigma^2)}{2r_\Sigma}}{\frac{47S_p(1-2S_p)^{\frac{-1}{2}}(1+C_1r^2)r_\Sigma}{2} - 47 - 54C_1r^2} \tag{45}$$

$$C_4 = \frac{c^2\left(\frac{2-S_p}{2+S_p}\right)}{(1+C_3r_\Sigma^2)(1+C_1r_\Sigma^2)^{\frac{7}{47}}} \tag{46}$$

The central redshift is

$$Z_0 = \frac{2S_p}{2-S_p} \tag{47}$$

The surface redshift is

$$Z_\Sigma = (1-2S_p)^{\frac{-1}{2}} - 1 \tag{48}$$

7. Conclusion

In the present article, we have obtained new anisotropic compact star model using variable separable form of metric components. Our model satisfy all the physical reality conditions. We matched the solution by joining the Schwarzschild metric at the boundary of the star. The metric potentials are free from any singularity at the centre, positive and finite inside the star. It has been observed that the physical parameters pressures, density, and redshift are positive at the centre and within the limit of realistic state equation and monotonically decreasing and the causality condition is obeyed throughout the fluid ball. Thus, the solution is well behaved for all values of Schwarzschild parameter S_p within the perfect fluid ball. Our solution is useful to construct the models of compact star like Gravitational lensing, Quark stars/Strange stars, Boson stars, Gravastars, Eternally Collapsing Objects and various high energy astronomical objects.

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