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PBIB-DESIGNS WITH $m = 1, 2, 3, 4$ AND $\lfloor \frac{p}{2} \rfloor$ ASSOCIATED CLASSES ARISING FROM γ_{cc}-SETS OF SOME SPECIAL KIND OF GRAPHS

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Dedicated to Prof. A.M. Mathai on his 80^{th} birth anniversary

Abstract: A set D of vertices of a connected graph $G = (V, E)$ is a co-connected dominating set if every vertex not in D is adjacent to some vertex in D and the subgraph induced $\langle V - D \rangle$ is connected. The co-connected domination number $\gamma_{cc}(G)$ is the minimum cardinality of a co-connected dominating set. A γ_{cc} - set is a minimum co-connected dominating set of G. In this paper, we obtain the Partially Balanced Incomplete Block (PBIB) - designs with m = 1, 2, 3, 4 and $\lfloor \frac{p}{2} \rfloor$ associated classes arising from γ_{cc} - sets of some special types of graphs.

Keywords and Phrases: Association schemes; Partially balanced incomplete block designs; Co-connected dominating sets.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected graph, without loops or multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. Any undefined term in this paper may be found in Harary [5].

Bose and Nair [3] introduced a class of binary, equireplicate and proper designs, which are called Partially Balanced Incomplete Block (PBIB)- Designs. This design is classified into different types on the basis of their association schemes. In brief, they can be grouped as PBIB(2)-Designs, PBIB(3)-Designs and higher associate-class PBIB-Designs. In each of the above mentioned groups there are further sub groups based on the types of the underlying association scheme.

Given ν objects a relation satisfying the following conditions is said to be an association scheme with m classes:

(i) Any two objects are either first associates, or second associates, ..., or m^{th} associates, the relation of association being symmetric.

(ii) Each object α has n_i i^{th} associates, the number n_i being independent of α.

(iii) If two objects α and β are i^{th} associates, then the number of objects which are j^{th} associates of α and k^{th} associates of β is p^i_{jk} and is independent of the i^{th} associates α and β. Also $p^i_{jk} = p^i_{kj}$.

With the association scheme on ν objects, the PBIB - Design is defined as follows.

The PBIB - Designs is an arrangement of ν objects into b sets (called blocks) of size k where $k < \nu$ such that

(i) Every object is contained in exactly r blocks.

(ii) Each block contains k distinct objects.

(iii) Any two objects which are i^{th} associates occur together in exactly λ_i blocks.

The numbers v, b, r, k, λ_1, λ_2, ..., λ_m are called the parameters of the first kind, whereas the numbers n_1, n_2, ..., n_m, p^i_{jk} $(i, j, k = 1, 2, \ldots, m)$ are called the parameters of the second kind.

Domination in graphs has been an extensively researched branch of graph theory. The concept of domination has existed and has been studied from a long time. Books on domination [8] has stimulated sufficient inspiration, leading to the expansive growth of this field. In a graph, a dominating set is a subset D of the vertices such that every vertex is either in D or is adjacent to a vertex in D. In [12], Sampathkumar and Walikar defined a connected dominating set D to be a dominating set D whose induced subgraph $\langle D\rangle$ is connected. Analogously, Kulli and Janakiram [10] initiates the concepts of Nonsplit domination, which we call Co-connected domination in the following sense. A dominating set D of a connected graph G is said to be co-connected dominating set if the induced subgraph $\langle V - D\rangle$ is connected. The co-connected domination number $\gamma_{cc}(G)$ is the minimum cardinality of a co-connected dominating set. This parameter is also known as outer connected domination and studied by [1] and [4]. For more details on connected domination and its related parameters, refer [9] and [13].

Harary et al., [6] and [7], considered the relation between isomorphic factorization of regular graphs and PBIB-Designs with 2 - association schemes. In [14], Walikar et al. have studied on the number of minimum dominating sets and PBIB-Designs arising from minimum dominating sets of paths and cycles. In [2] Anwar et al., have studied PBIB-Designs arising from minimal dominating sets of SRNT graphs. Analogously, PBIB-Designs and Association schemes arising from minimum connected dominating sets of a graph was studied by Manjunath et al., [11]. In this paper, we obtain the PBIB-Designs with $m = 1, 2, 3, 4$ and $\lfloor \frac{p}{2} \rfloor$ associated classes arising from minimum co-connected dominating sets (or, simply γ_{cc} - sets) of some special types of graphs.

2. Complete graphs with $m = 1$ Association Scheme

A complete graph K_p is a simple graph in which every pair of vertices is adjacent.

Theorem 2.1. *The collection of all* γ_{cc} *- sets of a complete graph* K_p *with* $p \geq 2$ *vertices form PBIB-Designs with 1-association scheme and parameters* $\nu = p$, $b = p$, $k = 1$, $r = 1$, $\lambda_1 = 0$.

Proof. Let a complete graph K_p be labeled as $v_1, v_2, \ldots, v_p$. The collection of all γ_{cc} - sets are given by $\{v_i\}$, $1 \leq i \leq p$. Two vertices u and v are 1^{st} associates if they are adjacent. For each vertex $v_i \in V(K_p)$, the vertices v_j, $1 \leq j \neq i \leq p$ are the 1^{st} associates. The parameters of second kind are given by $n_1 = p - 1$ and $P^1 = [p^1_{11}] = [p - 2]$ and the parameters of first kind are given by $\nu = p$, $b = p$, $k = 1$, $r = 1$, $\lambda_1 = 0$.

3. Hypercubes with $m = 1$, 2, 3 and 4 Association Schemes

The n-dimensional cube or hypercube Q_n is the simple graph whose vertices are the n-tuples with entries in 0, 1 and whose edges are the pairs of n-tuples that differ in exactly one position. Let Q_0 be a hypercube of dimension zero. Then the order of Q_0 is one. Clearly, for γ_{cc} - sets we consider only hypercube Q_n with $n \geq 1$. The interest in Hypercube has increased in the field of graph theory and has found its applications in the advent of massively parallel computers, whose structure results that of a Hypercube.

Theorem 3.1. *The collection of all* γ_{cc} *- sets of a hypercube* Q_1 *form PBIB-Designs with 1-association scheme and parameters* $\nu = 2$, $b = 2$, $k = 1$, $r = 1$, $\lambda_1 = 0$.

Proof. By Theorem 2.1, complete graph K_2 is isomorphic with Q_1. Thus the results follows.

Theorem 3.2. *The collection of all* γ_{cc} *- sets of a hypercube* Q_2 *form PBIB-Designs with 2-association scheme and parameters* $\nu = 4$, $b = 4$, $k = 2$, $r = 2$, $\lambda_1 = 1$, *and* $\lambda_2 = 0$.

Proof. Let a hypercube Q_2 be labeled as v_1, v_2, v_3 and v_4 with the collection of all

γ_{cc} - sets are given by $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$ and $\{v_4, v_1\}$. Then two vertices u and v are first associates if $d(u, v) = 1$ and second associates if there exists a vertex non-adjacent to any two of the adjacent vertices. Hence, the parameters of second kind are given by $n_1 = 2$ and the parameters of first kind are given by $\nu = 4$, $b = 4$, $k = 2$, $r = 2$, $\lambda_1 = 1$, and $\lambda_2 = 0$.

Theorem 3.3. *The collection of all γ_{cc} - sets of a hypercube Q_3 form PBIB-Designs with 3-association scheme and parameters $\nu = 8$, $b = 4$, $k = 2$, $r = 1$, $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 1$.*

Proof. Let a hypercube Q_3 be labeled as v_1, v_2, ..., v_8, as shown in Figure−1,

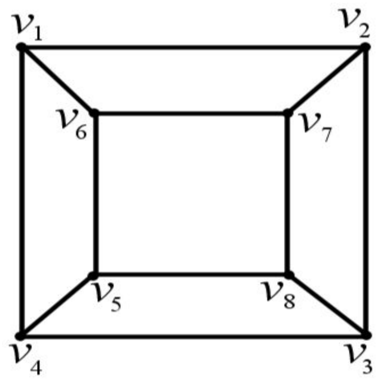

Figure 1: Hypercube Q_3

with the collection of all γ_{cc} - sets, given by $\{v_2, v_5\}$, $\{v_3, v_6\}$, $\{v_4, v_7\}$ and $\{v_1, v_8\}$.

Then the two distinct vertices u and v are said to be first associates if $d(u, v) = 1$; second associates if $d(u, v) = 2$ and third associates if $d(u, v) = 3$. Hence, the first, second and third associates of each vertex are shown in the following table.

vertex	1^{st}	2^{nd}	3^{rd}
v_1	v_2, v_4, v_6	v_3, v_5, v_7	v_8
v_2	v_1, v_3, v_7	v_4, v_6, v_8	v_5
v_3	v_2, v_4, v_8	v_1, v_5, v_7	v_6
v_4	v_1, v_3, v_5	v_2, v_6, v_8	v_7
v_5	v_4, v_6, v_8	v_1, v_3, v_7	v_2
v_6	v_1, v_5, v_7	v_2, v_4, v_8	v_3
v_7	v_2, v_6, v_8	v_1, v_3, v_5	v_4
v_8	v_3, v_5, v_7	v_2, v_4, v_6	v_1

With this association, the parameters of second kind are given by $n_1 = 3$, $n_2 = 3$, $n_3 = 1$ and

$$P^1 = \begin{pmatrix} P^1_{11} & P^1_{12} & P^1_{13} \\ P^1_{21} & P^1_{22} & P^1_{23} \\ P^1_{31} & P^1_{32} & P^1_{33} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} P^2_{11} & P^2_{12} & P^2_{13} \\ P^2_{21} & P^2_{22} & P^2_{23} \\ P^2_{31} & P^2_{32} & P^2_{33} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} P^3_{11} & P^3_{12} & P^3_{13} \\ P^3_{21} & P^3_{22} & P^3_{23} \\ P^3_{31} & P^3_{32} & P^3_{33} \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the parameters of first kind are given by $\nu = 8$, $b = 4$, $k = 2$, $r = 1$, $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 1$.

Theorem 3.4. *The collection of all γ_{cc} - sets of a hypercube Q_4 form PBIB-Designs with 4-association scheme and parameters $\nu = 16$, $b = 40$, $k = 4$, $r = 10$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 4$ and $\lambda_4 = 4$.*

Proof. Let a hypercube Q_4 be labeled as $v_1, v_2, \ldots, v_{16}$, as shown in Figure−2,

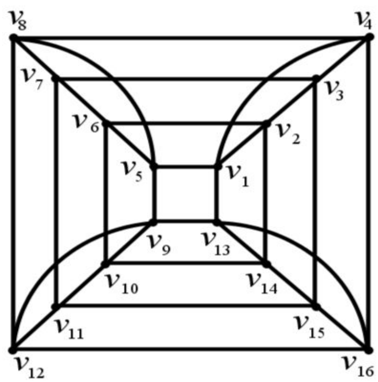

Figure 2: Hypercube Q_4

with the set of all γ_{cc} - sets are given by
$\{v_1, v_3, v_{10}, v_{12}\}$, $\{v_5, v_7, v_{14}, v_{16}\}$, $\{v_9, v_{11}, v_2, v_4\}$, $\{v_{13}, v_{15}, v_6, v_8\}$,
$\{v_3, v_4, v_9, v_{10}\}$, $\{v_7, v_8, v_{13}, v_{14}\}$, $\{v_{11}, v_{12}, v_1, v_2\}$, $\{v_{15}, v_{16}, v_5, v_6\}$,
$\{v_2, v_3, v_9, v_{12}\}$, $\{v_6, v_7, v_{13}, v_{16}\}$, $\{v_{10}, v_{11}, v_1, v_4\}$, $\{v_{14}, v_{15}, v_5, v_8\}$,
$\{v_1, v_{13}, v_7, v_{11}\}$, $\{v_{13}, v_9, v_7, v_3\}$, $\{v_9, v_5, v_3, v_{15}\}$, $\{v_5, v_1, v_{15}, v_{11}\}$,

$\{v_2, v_{14}, v_8, v_{12}\}$, $\{v_{14}, v_{10}, v_4, v_8\}$, $\{v_{10}, v_6, v_4, v_{16}\}$, $\{v_6, v_2, v_{12}, v_{16}\}$,
$\{v_8, v_3, v_{13}, v_{10}\}$, $\{v_4, v_{15}, v_9, v_6\}$, $\{v_{16}, v_{11}, v_5, v_2\}$, $\{v_{12}, v_7, v_1, v_{14}\}$,
$\{v_4, v_{14}, v_9, v_7\}$, $\{v_{16}, v_{10}, v_5, v_3\}$, $\{v_{12}, v_6, v_1, v_{15}\}$, $\{v_8, v_2, v_{13}, v_{11}\}$,
$\{v_7, v_2, v_{16}, v_9\}$, $\{v_3, v_{14}, v_{12}, v_5\}$, $\{v_{15}, v_{10}, v_8, v_1\}$, $\{v_{11}, v_6, v_4, v_{13}\}$,
$\{v_6, v_3, v_{13}, v_{12}\}$, $\{v_2, v_{15}, v_9, v_8\}$, $\{v_{14}, v_{11}, v_5, v_4\}$, $\{v_{10}, v_7, v_1, v_{16}\}$,
$\{v_7, v_1, v_{15}, v_9\}$, $\{v_3, v_{13}, v_{11}, v_5\}$, $\{v_8, v_2, v_{16}, v_{10}\}$, $\{v_4, v_{14}, v_{12}, v_6\}$.
Then two vertices u and v are said to be first associated if $d(u, v) = 1$, second associated if $d(u, v) = 2$, third associated if $d(u, v) = 3$ and fourth associated if $d(u, v) = 4$.
With this association, the parameters of second kind are given by $n_1 = 4$, $n_2 = 6$, $n_3 = 4$, $n_4 = 1$ and

$$P^1 = \begin{pmatrix} P^1_{11} & P^1_{12} & P^1_{13} & P^1_{14} \\ P^1_{21} & P^1_{22} & P^1_{23} & P^1_{24} \\ P^1_{31} & P^1_{32} & P^1_{33} & P^1_{34} \\ P^1_{41} & P^1_{42} & P^1_{43} & P^1_{44} \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} P^2_{11} & P^2_{12} & P^2_{13} & P^2_{14} \\ P^2_{21} & P^2_{22} & P^2_{23} & P^2_{24} \\ P^2_{31} & P^2_{32} & P^2_{33} & P^2_{34} \\ P^2_{41} & P^2_{42} & P^2_{43} & P^2_{44} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} P^3_{11} & P^3_{12} & P^3_{13} & P^3_{14} \\ P^3_{21} & P^3_{22} & P^3_{23} & P^3_{24} \\ P^3_{31} & P^3_{32} & P^3_{33} & P^3_{34} \\ P^3_{41} & P^3_{42} & P^3_{43} & P^3_{44} \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} P^4_{11} & P^4_{12} & P^4_{13} & P^4_{14} \\ P^4_{21} & P^4_{22} & P^4_{23} & P^4_{24} \\ P^4_{31} & P^4_{32} & P^4_{33} & P^4_{34} \\ P^4_{41} & P^4_{42} & P^4_{43} & P^4_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 6 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the parameters of first kind are given by $\nu = 16$, $b = 40$, $k = 4$, $r = 10$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 4$ and $\lambda_4 = 4$.

In this section, we pose a following Open problem.
Open Problem: Find the collection of all γ_{cc} - sets of a hypercube Q_n with $k \geq 5$ and form PBIB-Designs with $m \geq 5$-association schemes (if possible generalize).

4. Cycles with $m = \lfloor \frac{p}{2} \rfloor$ Association Schemes

A cycle of length p is the graph C_p on $p \geq 3$ vertices $v_1, v_2, \ldots, v_{p-1}$ with p

edges $(v_1, v_2), (v_2, v_3), \cdots, (v_p, v_1)$.

Theorem 4.1 *The collection of all γ_{cc} - sets of a cycle C_p, $p \geq 3$ forms a PBIB-Designs with $\lfloor \frac{p}{2} \rfloor$ association scheme and parameters $\nu = p$, $b = p$, $k = p - 2$, $r = p - 2$, $\lambda_1 = p - 3$ and $\lambda_i = p - 4$, $2 \leq i \leq \lfloor \frac{p}{2} \rfloor$.*

Proof. Let a cycle C_p be labeled as $v_1, v_2, \ldots, v_p$ with the collection of all γ_{cc} - sets are given by $V(C_p) - \{v_i, v_{i+1 \ (mod\ p)}\}$, $1 \leq i \leq p$. Then two vertices v_i, v_j are said to be k^{th} $(1 \leq k \leq \lfloor \frac{p}{2} \rfloor)$ associated if $d(v_i, v_j) = k$. Thus the following cases are arise:

Case 1. If p is even, then the 1^{st}, 2^{nd}, 3^{rd}, ..., $\frac{p}{2}^{th}$ associates of each vertex are given by

vertex	1^{st} associates	2^{nd} associates	k^{th} associates	$\frac{p}{2}^{th}$ associates
v_1	v_p, v_2	v_{p-1}, v_3	$v_{p-(k-1)}$, v_{1+k}	$v_{1+\frac{p}{2}}$
v_2	v_1, v_3	v_p, v_4	$v_{p-(k-2)}$, v_{2+k}	$v_{2+\frac{p}{2}}$
v_3	v_2, v_4	v_1, v_5	$v_{p-(k-3)}$, v_{3+k}	$v_{3+\frac{p}{2}}$
.	.	.	.	.
.	.	.	.	.
v_i	v_{i-1}, v_{i+1}	v_{i-2}, v_{i+2}	$v_{p-(k-i)}$, v_{i+k}	$v_{i+\frac{p}{2}(mod\ p)}$
.	.	.	.	.
.	.	.	.	.
v_p	v_{p-1}, v_1	v_{p-2}, v_2	v_{p-k}, v_k	$v_{\frac{p}{2}}$

With this association scheme, the parameters of second kind are given by $n_i = 2$, $1 \leq i \leq \frac{p}{2} - 1$ and $n_{\frac{p}{2}} = 1$ and

$$P_{ij}^k = \begin{pmatrix} P_{11}^k & P_{12}^k & . & . & . & P_{1\ \frac{p}{2}}^k \\ P_{21}^k & P_{22}^k & . & . & . & P_{2\ \frac{p}{2}}^k \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ P_{\frac{p}{2}1}^k & P_{\frac{p}{2}2}^k & . & . & . & P_{\frac{p}{2}\ \frac{p}{2}}^k \end{pmatrix}$$

with entries given as follows,

For $k = 1$, $P_{ij}^1 = 1$ for $1 \leq i \leq \frac{p}{2} - 1$, $j = i + 1$

$P_{ij}^1 = 1$ for $i = 1 + l$, $1 \leq l \leq \frac{p}{2} - 1$, $j = l$

For $2 \leq k \leq r - 2$,

$P_{ij}^k = 1$ for $1 \leq i \leq k - 1$ and $j = k - i$

$P_{ij}^k = 1$ for $1 \le i \le r-k$ and $j = k+i$
$P_{ij}^k = 1$ for $i = r-k+l$, $1 \le l \le k$, $j = r-l$
$P_{ij}^k = 1$ for $i = k+l$, $1 \le l \le r-1-k$, $j = l$
For $k = r-1$,
$P_{ij}^k = 1$ for $1 \le i \le k-1$ and $j = k-i$
$P_{ij}^k = 1$ for $1 \le i \le r-k$ and $j = k+i$
$P_{ij}^k = 1$ for $i = r-k+l$, $1 \le l \le k$, $j = r-l$
For $k = r$,
$P_{ij}^k = 2$, for $1 \le i \le \frac{p-2}{2}$, $j = r-i$
and the other entries are all zero. Hence the parameters of first kind are given by $\nu = p$, $b = p$, $k = p-2$, $r = p-2$, $\lambda_1 = p-3$ and $\lambda_i = p-4$, $2 \le i \le \frac{p}{2}$.

Case 2. If p is odd, then the 1^{st}, 2^{nd}, 3^{rd}, $\ldots$, $\frac{p-1}{2}^{th}$ associates of each vertex are given by

vertex	1^{st}	2^{nd}	k^{th}	$\frac{p-1}{2}^{th}$ associates
v_1	$v_p,\ v_2$	$v_{p-1},\ v_3$	$v_{p-(k-1)},\ v_{1+k}$	$v_{1+\frac{p-1}{2}},\ v_{1+\frac{p-1}{2}+1}$
v_2	$v_1,\ v_3$	$v_p,\ v_4$	$v_{p-(k-2)},\ v_{2+k}$	$v_{2+\frac{p-1}{2}},\ v_{2+\frac{p-1}{2}+1}$
v_3	$v_2,\ v_4$	$v_1,\ v_5$	$v_{p-(k-3)},\ v_{3+k}$	$v_{3+\frac{p-1}{2}},\ v_{3+\frac{p-1}{2}+1}$
.	.	.	.	.
.	.	.	.	.
v_i	$v_{i-1},\ v_{i+1}$	$v_{i-2},\ v_{i+2}$	$v_{p-(k-i)},\ v_{i+k}$	$v_{i+\frac{p-1}{2}(mod\ p)},\ v_{i+\frac{p-1}{2}+1(mod\ p)}$
.	.	.	.	.
.	.	.	.	.
v_p	$v_{p-1},\ v_1$	$v_{p-2},\ v_2$	$v_{p-k},\ v_k$	$v_{\frac{p-1}{2}},\ v_{\frac{p-1}{2}+1}$

With this association scheme, the parameters of second kind are given by $n_i = 2$, $1 \le i \le \frac{p-1}{2}$ and

$$P_{ij}^k = \begin{pmatrix} P_{11}^k & P_{12}^k & . & . & . & P_{1\ \frac{p-1}{2}}^k \\ P_{21}^k & P_{22}^k & . & . & . & P_{2\ \frac{p-1}{2}}^k \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ P_{\frac{p}{2}1}^k & P_{\frac{p}{2}2}^k & . & . & . & P_{(\frac{p-1}{2})\ (\frac{p-1}{2})}^k \end{pmatrix}$$

with entries given as follows,
For $k = 1$,

$P^1_{ij} = 1$ for $1 \leq i \leq \frac{p-1}{2} - 1$, $j = i + 1$
$P^1_{ij} = 1$ for $i = 1 + l$, $1 \leq l \leq \frac{p-1}{2} - 1$, $j = l$
$P^1_{ij} = 1$ for $i = r$ and $j = r$
For $2 \leq k \leq r - 1$,
$P^k_{ij} = 1$ for $1 \leq i \leq k - 1$ and $j = k - i$
$P^k_{ij} = 1$ for $1 \leq i \leq r - k$ and $j = k + i$
$P^k_{ij} = 1$ for $i = r - k + l$, $1 \leq l \leq k$, $j = r - (l - 1)$
$P^k_{ij} = 1$ for $i = k + l$, $1 \leq l \leq r - k$, $j = l$
For $k = r$,
$P^k_{ij} = 1$, for $1 \leq i \leq r - 1$, $j = r - i$
$P^k_{ij} = 1$, for $i = 1 + l$, $0 \leq l \leq r - 1$, $j = r - l$ and the other entries are all zero.
Hence the parameters of first kind are given by $\nu = p$, $b = p$, $k = p - 2$, $r = p - 2$, $\lambda_1 = p - 3$ and $\lambda_i = p - 4$, $2 \leq i \leq \frac{p-1}{2}$.

For illustration of the above Theorem, we consider the following:

In the Case 1, if we consider an even cycle C_{10}, labeled as v_1, v_2, ..., v_{10}, then all γ_{cc} - sets of C_{10} are given by
$\{v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, $\{v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_1\}$,
$\{v_5, v_6, v_7, v_8, v_9, v_{10}, v_1, v_2\}$, $\{v_6, v_7, v_8, v_9, v_{10}, v_1, v_2, v_3\}$,
$\{v_7, v_8, v_9, v_{10}, v_1, v_2, v_3, v_4\}$, $\{v_8, v_9, v_{10}, v_1, v_2, v_3, v_4, v_5\}$,
$\{v_9, v_{10}, v_1, v_2, v_3, v_4, v_5, v_6\}$, $\{v_{10}, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$,
$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$, $\{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$.
Here, the 1^{st}, 2^{nd}, 3^{rd}, 4^{th} and 5^{th} associates of each vertex are given by in the following table

vertex	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
v_1	v_{10}, v_2	v_9, v_3	v_8, v_4	v_7, v_5	v_6
v_2	v_1, v_3	v_{10}, v_4	v_9, v_5	v_8, v_6	v_7
v_3	v_2, v_4	v_1, v_5	v_{10}, v_6	v_9, v_7	v_8
v_4	v_3, v_5	v_2, v_6	v_1, v_7	v_{10}, v_8	v_9
v_5	v_4, v_6	v_3, v_7	v_2, v_8	v_1, v_9	v_{10}
v_6	v_5, v_7	v_4, v_8	v_3, v_9	v_2, v_{10}	v_1
v_7	v_6, v_8	v_5, v_9	v_4, v_{10}	v_3, v_1	v_2
v_8	v_7, v_9	v_6, v_{10}	v_5, v_1	v_4, v_2	v_3
v_9	v_8, v_{10}	v_7, v_1	v_6, v_2	v_5, v_3	v_4
v_{10}	v_9, v_1	v_8, v_2	v_7, v_3	v_6, v_4	v_5

and

$$P^1 = \begin{pmatrix} P^1_{11} & P^1_{12} & P^1_{13} & P^1_{14} & P^1_{15} \\ P^1_{21} & P^1_{22} & P^1_{23} & P^1_{24} & P^1_{25} \\ P^1_{31} & P^1_{32} & P^1_{33} & P^1_{34} & P^1_{35} \\ P^1_{41} & P^1_{42} & P^1_{43} & P^1_{44} & P^1_{45} \\ P^1_{51} & P^1_{52} & P^1_{53} & P^1_{54} & P^1_{55} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} P^2_{11} & P^2_{12} & P^2_{13} & P^2_{14} & P^2_{15} \\ P^2_{21} & P^2_{22} & P^2_{23} & P^2_{24} & P^2_{25} \\ P^2_{31} & P^2_{32} & P^2_{33} & P^2_{34} & P^2_{35} \\ P^2_{41} & P^2_{42} & P^2_{43} & P^2_{44} & P^2_{45} \\ P^2_{51} & P^2_{52} & P^2_{53} & P^2_{54} & P^2_{55} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} P^3_{11} & P^3_{12} & P^3_{13} & P^3_{14} & P^3_{15} \\ P^3_{21} & P^3_{22} & P^3_{23} & P^3_{24} & P^3_{25} \\ P^3_{31} & P^3_{32} & P^3_{33} & P^3_{34} & P^3_{35} \\ P^3_{41} & P^3_{42} & P^3_{43} & P^3_{44} & P^3_{45} \\ P^3_{51} & P^3_{52} & P^3_{53} & P^3_{54} & P^3_{55} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} P^4_{11} & P^4_{12} & P^4_{13} & P^4_{14} & P^4_{15} \\ P^4_{21} & P^4_{22} & P^4_{23} & P^4_{24} & P^4_{25} \\ P^4_{31} & P^4_{32} & P^4_{33} & P^4_{34} & P^4_{35} \\ P^4_{41} & P^4_{42} & P^4_{43} & P^4_{44} & P^4_{45} \\ P^4_{51} & P^4_{52} & P^4_{53} & P^4_{54} & P^4_{55} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P^5 = \begin{pmatrix} P^5_{11} & P^5_{12} & P^5_{13} & P^5_{14} & P^5_{15} \\ P^5_{21} & P^5_{22} & P^5_{23} & P^5_{24} & P^5_{25} \\ P^5_{31} & P^5_{32} & P^5_{33} & P^5_{34} & P^5_{35} \\ P^5_{41} & P^5_{42} & P^5_{43} & P^5_{44} & P^5_{45} \\ P^5_{51} & P^5_{52} & P^5_{53} & P^5_{54} & P^5_{55} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the Case 2, if we consider an odd cycle C_9, labeled as v_1, v_2, ..., v_9, then all γ_{cc} - sets of C_9 are given by
$\{v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$, $\{v_4, v_5, v_6, v_7, v_8, v_9, v_1\}$,
$\{v_5, v_6, v_7, v_8, v_9, v_1, v_2\}$, $\{v_6, v_7, v_8, v_9, v_1, v_2, v_3\}$,
$\{v_7, v_8, v_9, v_1, v_2, v_3, v_4\}$, $\{v_8, v_9, v_1, v_2, v_3, v_4, v_5\}$,
$\{v_9, v_1, v_2, v_3, v_4, v_5, v_6\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$,
$\{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$.
Here, the 1^{st}, 2^{nd}, 3^{rd}, 4^{th} associates of each vertex are given in the following table:

vertex	1^{st}	2^{nd}	3^{rd}	4^{th}
v_1	v_9, v_2	v_8, v_3	v_7, v_4	v_6, v_5
v_2	v_1, v_3	v_9, v_4	v_8, v_5	v_7, v_6
v_3	v_2, v_4	v_1, v_5	v_9, v_6	v_8, v_7
v_4	v_3, v_5	v_2, v_6	v_1, v_7	v_9, v_8
v_5	v_4, v_6	v_3, v_7	v_2, v_8	v_1, v_9
v_6	v_5, v_7	v_4, v_8	v_3, v_9	v_2, v_1
v_7	v_6, v_8	v_5, v_9	v_4, v_1	v_3, v_2
v_8	v_7, v_9	v_6, v_1	v_5, v_2	v_4, v_3
v_9	v_8, v_1	v_7, v_2	v_6, v_3	v_5, v_4

and

$$P^1 = \begin{pmatrix} P^1_{11} & P^1_{12} & P^1_{13} & P^1_{14} \\ P^1_{21} & P^1_{22} & P^1_{23} & P^1_{24} \\ P^1_{31} & P^1_{32} & P^1_{33} & P^1_{34} \\ P^1_{41} & P^1_{42} & P^1_{43} & P^1_{44} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} P^2_{11} & P^2_{12} & P^2_{13} & P^2_{14} \\ P^2_{21} & P^2_{22} & P^2_{23} & P^2_{24} \\ P^2_{31} & P^2_{32} & P^2_{33} & P^2_{34} \\ P^2_{41} & P^2_{42} & P^2_{43} & P^2_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} P^3_{11} & P^3_{12} & P^3_{13} & P^3_{14} \\ P^3_{21} & P^3_{22} & P^3_{23} & P^3_{24} \\ P^3_{31} & P^3_{32} & P^3_{33} & P^3_{34} \\ P^3_{41} & P^3_{42} & P^3_{43} & P^3_{44} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} P^4_{11} & P^4_{12} & P^4_{13} & P^4_{14} \\ P^4_{21} & P^4_{22} & P^4_{23} & P^4_{24} \\ P^4_{31} & P^4_{32} & P^4_{33} & P^4_{34} \\ P^4_{41} & P^4_{42} & P^4_{43} & P^4_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

References

[1] M. H. Akhbari, R. Hasni, O. Favaron, H. Karami and S. M. Sheikholeshami, On the outer-connected domination in graphs, *J. Comb. Optim.*, **213(1)**, (2013), 10-18.

[2] Anwar Alwardi and N. D. Soner, Partially balanced incomplete block designs arising from some minimal dominating sets of SRNT graphs, *International Journal of Mathematical Archive,* **2(2)** (2011), 233-235.

[3] R. C. Bose and K. R. Nair, Partially balanced incomplete block designs, Sankhya, **4** (1939), 337-372.

[4] J. Cyman, The outer-connected domination number of a graph, *Australasian J. of Combinatorics,* **28** (2007), 35-46.

[5] F. Harary, Graph Theory, Addison-Wesley, Reading Mass (1969).

[6] F. Harary, R. W. Robinson and N. C. Wormald, Isomorphic factorization I: Complete Graphs, *Trans. Amer. Math. Soc.,* **242** (1978), 243-260.

[7] F. Harary and W. D. Wallis, Isomorphic factorization-II: Combinatorial designs, Proc. Of the Eigth Southeastern Conference on Combinatorics, Graph Theory and Computing, *Utilitas Math,* Pub, Winnipeg, (1978), 13-28.

[8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in graphs, Marcel Dekker, Inc, New York, 1998.

[9] S. T. Hedetniemi and R. Laskar, Connected domination in graphs, *Graph theory and Combinatorics,* Academic Press, London, (1984), 209-217.

[10] V. R. Kulli and B. Janakiram, The nonsplit domination number of a graph, *Indian J. Pure Appl. Math.,* **31(4)** (2000), 441-447.

[11] N. Manjunath, G. Deepak and B. Chaluvaraju, PBIB-Designs and Association schemes arising from minimum connected dominating sets of a graph, South Asian J. of Maths. **4(5)** (2014), 225-232.

[12] E. Sampathkumar and H. B. Walikar, The connected domination number of a graph, *J. Math. Phys. Sci.,* **13** (1979), 607-613.

[13] N. D. Soner, B. Janakiram and B. Chaluvaraju, The k-nonsplit domination number of a graph, *International J. of Mgmt. and Syst.,* **22(1)** (2006), 53-58.

[14] H. B. Walikar, H. S. Ramane, B. D. Acharya, H. S. Shekarappa and S. Arumugam, Partially balanced incomplete block design arising from minimum dominating sets of paths and cycles,*AKCE J. Graphs Combin.,* **4(2)** (2007), 223-232.