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ON THE LINEAR COMBINATION OF N LOGISTIC RANDOM VARIABLES AND RELIABILITY ANALYSIS

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Dedicated to Prof. A.M. Mathai on his 80^{th} birth anniversary

Abstract: Logistic distributions have been widely applied to model data in both pure and applied sciences. In the present paper, the probability density and cumulative distribution functions of the linear combination of N independent and not identically distributed logistic random variables have been obtained in terms of the H-function. By means of the latter, reliability measures of the type $P(\sum_{i=1}^{N_1} X_i < \sum_{j=1}^{N_2} Y_j)$, when X_i, $i = 1, ..., N_1$ and Y_j, $j = 1, ..., N_2$ are logistic random variables have been derived. Also, a highly accurate approximated expression has been built for the case $N_1 = N_2 = 1$ by means of curve fitting techniques, avoiding the need for H-function calculations in this case. Numerical experiments have been carried out, revealing that the expressions proposed correctly predicted the reliability measures considered.

Keywords and Phrases: logistic distribution; linear combination; reliability analysis.

2010 Mathematics Subject Classification: 33C60; 97K80.

1. Introduction

Stochastic formulations are fundamental to the modeling of natural phenomena. In the last decades, the importance of the logistic distribution has been widely recognized; placing this type of random variable among the most studied ones. Also, not only pure, but also applied scientists have found in this distribution a very good model for predicting their variables of interest. One of the remarkable properties of the logistic distribution is that it closely approximates normal distributions. This feature has been extensively used since the former is invertible, while the latter is not. This way, consider a random variable W. Let W follow a logistic distribution with mean $\mu \in R$ and scale parameter $\sigma > 0$. One says $W \sim L(\mu, \sigma)$ and both the probability density function (p.d.f) and the cumulative distribution function (c.d.f.) of W are given by

$$f(x;\sigma,\mu) = \frac{e^{-(\frac{x-\mu}{\sigma})}}{\sigma[1+e^{-(\frac{x-\mu}{\sigma})}]^2}, \quad \forall x \in R, \tag{1}$$

and

$$F(x;\sigma,\mu) = \frac{1}{1+e^{-(\frac{x-\mu}{\sigma})}}, \quad \forall x \in \mathbb{R}, \tag{2}$$

respectively.

In general, while dealing with random variables, defining algebraic operations over them is of great interest. In special, the algebra of random variables shows how to obtain the distributions of product, ratio, sum and difference of random variables.

Over the last half century, due to the high applicability of the logistic distribution, special attention has been given to the obtention of the distribution of the sum of logistic random variables. Goel (1975) presented a slow-converging formula for the cumulative distribution function of the sum of independent and identically distributed (i.i.d) logistic random variables with mean zero and scale parameter 1. On the other hand, George and Mudholkar (1983) discussed improvements on the expressions in (Goel, 1975) by proposing other series representations. The formulation presented in (George and Mudholkar, 1983) consisted basically of triple series, which require considerable computational effort to evaluate.

Ojo and Adeyemi (1989) obtained the distribution function of the sum of i.i.d generalized logistic distribution in terms of infinite series. In Ojo and Adeyemi (1989) approximations in terms of student$-t$ distributions have also been presented, but as in the case of the expressions from (George and Mudholkar , 1983), the computational efforts required to evaluate the formulas were enormous.

More recently, Ojo (2002) proposed approximations to the expressions in (Ojo and Adeyemi , 1989) by considering the mixture of normal, logistic and double exponential distributions. Also, Ojo (2003) discusses alternative approaches to obtain the distribution function of the sum of i.i.d generalized logistic distribution. In (Ojo, 2003), alternative representations of the distribution function are given in terms of derivatives and series.

Even though considerable work has been done on the subject, a closed-form compact representation for the sum of independent not identically distributed logistic random variables has not been given. In the present paper, the linear combination of logistic random variables with different means and scale parameters is given, in a compact form, in terms of the H function. This latter function is a generalized hypergeometric special function whose importance has been widely recognized. In special, Springer (1979) discusses the central role of this function to the study of the algebra of random variables.

Besides the pure statistical applications where the linear combination itself is sought, reliability models can also be derived based on the latter.

For example, if X is the strength of the component of some system which is subject to a stress Y, $R = P(X < Y) = P(X - Y < 0)$ is a measure of component reliability. The evaluation and the estimate of R for a set of data is called in statistical quality control of stress-strength problem. A recent review on this topic is given in (Kotz et al., 2003). When X and Y are assumed to be independent and identically distributed (iid) random variables, the stress-strength problem have been extensively studied by many authors. Among the distributions considered are normal, exponential, gamma, Weibull, Pareto, logistic and extreme value family.

In general, mathematical procedures are used to estimate R. For example, Al-Mutairi et al. (2011) used maximum likelihood and bootstrap methods to derive confidence intervals for R when the considered random variables are of exponential type. Recently, a new exponential-type distribution has been introduced in (Lemonte, 2013), being estimates for the reliability measure R given when both random variables compared belong to this new type of distribution family. Also, Rezaei et al. (2010) and Krishnamoorthy and Lin (2010) proposed estimates for R when both the distributions compared were generalized Pareto and Weibull, respectively.

Regarding logistic-type distributions, Asgharzadeh et al. (2013) derived estimators for R when both the distributions compared are generalized logistic. Haghighi and Shayib (2010) studied the reliability R for independent random variables X and Y following logistic distributions with null location parameters. One of the purposes of this note is to examine R when both the two variables are logistics with non-null scale and different location parameters. An exact expression for R is derived in terms of the H-function. Also, an approximation to R, in terms of elementary functions, is proposed. The latter is based on curve fitting methods, as shall be discussed later on in the present paper.

It is worth noticing that the main purpose of the present paper is to provide exact and approximate formulas for reliability measures $R = P(\sum_{i=1}^{N_1} X_i < \sum_{j=1}^{N_2} Y_j)$, when X_i, $i = 1, ..., N_1$ and Y_j, $j = 1, ..., N_2$ are logistic random variables. Thus, this paper is to be used as a theoretical basis for future accurate practical applications.

Since the analytical expressions are derived in terms of the H-function, in order to familiarize the reader, a few definitions are given in the next section.

2. H-function

The H - function (see (Mathai et al., 2010)) is defined, as a contour complex integral which contains gamma functions in their integrands, by

$$H_{p,q}^{m,n}\left[z \left| \begin{array}{cccccc} (a_1, A), & \ldots, & (a_n, A_n), & (a_{n+1}, A_{n+1}), & \ldots, & (a_p, A_p) \\ (b_1, B_1), & \ldots, & (b_m, B_m), & (b_{m+1}, B_{m+1}), & \ldots, & (b_q, B_q) \end{array} \right.\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\prod_{j=n+1}^{p}\Gamma(a_j + A_j s)} z^{-s} ds, \tag{3}$$

where A_j and B_j are assumed to be positive quantities and all the a_j and b_j may be complex. The contour L runs from $c - i\infty$ to $c + i\infty$ such that the poles of $\Gamma(b_j + B_j s)$, $j = 1, \ldots, m$ lie to the left of L and the poles of $\Gamma(1 - a_j - A_j s)$, $j = 1, \ldots, n$ lie to the right of L.

The Mellin transform of the H -function is given by

$$\int_0^\infty x^{s-1} H_{p,q}^{m,n}\left[cx \left| \begin{array}{c} (a_p, A_p) \\ (b_q, B_q) \end{array} \right.\right] dx = \frac{c^{-s}\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\prod_{j=n+1}^{p}\Gamma(a_j + A_j s)} z^{-s}. \tag{4}$$

Given these few expressions, one shall proceed to obtain the p.d.f and c.d.f of the linear combination of logistic random variables by means of the H-function.

3. The Linear Combination of N Logistic Random Variables

In this section a brief description of the problem and its full solutions are discussed.

3.1 Problem Statement

Let $X_i \sim L(\mu_i, \sigma_i)$. Then, one seeks the probability density function of the random variable Z, such that:

$$Z = \sum_{i=1}^{N} b_i X_i \tag{5}$$

in which b_i, $i = 1, ..., N$ are real numbers.

At first, one may notice that if $X_i \sim L(\mu_i, \sigma_i)$, then $b_i X_i \sim L(b_i\mu_i, \sigma_i|b_i|)$. This can be readily verified by noticing that:

$$P(b_i X_i \leq y) = \begin{cases} P(X_i \leq y/b_i) = F(y, b_i\sigma_i, b_i\mu_i) & b_i > 0 \\ P(X_i > y/b_i) = F(y, -b_i\sigma_i, b_i\mu_i) & b_i < 0 \end{cases} \tag{6}$$

This way, by using (5) and (6), the variable Z can be rewritten as the sum of the logistic variables $Y_i, Y_i = b_i X_i \sim L(b_i\mu_i, \sigma_i|b_i|)$. In order to provide a closed form exact representation for the probability density function of the random variable Z, one shall first proceed to the obtention of the characteristic functions of the random variables Y_i, $i = 1, ..., N$.

3.2 The Characteristic Function of a Logistic Random Variable

The characteristic function of a given random variable is nothing but the Fourier transform of its probability density function. The characteristic function of a logistic random variables is widely known in the literature. On the other hand, for completeness of study, such characteristic function is obtained in detail in the present paper. This way, by means of (1), the characteristic function $\phi_i(t)$ (CF) for the random variables Y_i, $i = 1, ..., N$ can be given as:

$$\phi_i(t) = \int_{-\infty}^{\infty} e^{jtx} \frac{e^{-\frac{(x-\mu_i b_i)}{\sigma_i|b_i|}}}{\sigma_i|b_i| \left(1 + e^{-\frac{(x-\mu_i b_i)}{\sigma_i|b_i|}}\right)^2} dx, \tag{7}$$

where $j = \sqrt{-1}$.

By means of the variable change $r = e^{-\frac{(x-\mu_i b_i)}{\sigma_i|b_i|}} / \left(1 + e^{-\frac{(x-\mu_i b_i)}{\sigma_i|b_i|}}\right)$, the integral in

(7) becomes:

$$\phi_i(t) = e^{jtb_i\mu_i} \int_0^1 r^{-jt\sigma_i|b_i|}(1-r)^{jt\sigma_i|b_i|}dr. \quad (8)$$

One may consider the beta function, defined as:

$$B(a,b) = \int_0^1 r^{a-1}(1-r)^{b-1|}dr = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad (9)$$

in which $\Gamma(.)$ denotes the gamma function.

This way, (8) and (9) provide:

$$\phi_i(t) = e^{jtb_i\mu_i}B(1-jt\sigma_i|b_i|, 1+jt\sigma_i|b_i|) = e^{jtb_i\mu_i}\Gamma(1-jt\sigma_i|b_i|)\Gamma(1+jt\sigma_i|b_i|). \quad (10)$$

It is widely known that the characteristic function of the sum of independent random variables is the product of the individual characteristic functions (Springer , 1979). Thus, the CF of the random variable Z, $\phi_Z(t)$, can be given as:

$$\phi_Z(t) = \prod_{i=1}^{N} \left[e^{jtb_i\mu_i}\Gamma(1-jt\sigma_i|b_i|)\Gamma(1+jt\sigma_i|b_i|)\right]. \quad (11)$$

In order to obtain the probability density function of the random variable Z, one shall invert the Fourier transform applied to obtain the CF. This process is described in the next subsection.

3.3 The Probability Density Function of the Linear Combination of Logistic Random Variables

Being the CF of the random variable Z known, by means of the inversion formula for Fourier transform, one shall get that the probability density function of Z, $f_Z(x)$, as:

$$f_Z(x;\overline{\sigma},\overline{\mu},\overline{b}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jt\left(x-\sum\limits_{i=1}^{N} b_i\mu_i\right)} \prod_{i=1}^{N}\left[\Gamma(1-jt\sigma_i|b_i|)\Gamma(1+jt\sigma_i|b_i|)\right]dt, \quad (12)$$

where $\overline{\sigma}$, $\overline{\mu}$ and $\overline{b}$ represent the vectors of means, scale parameters and coefficients, respectively.

It is possible to transform the real integral in (12) into a contour integral in the complex plane. In order to do that, let one consider the variable change $jt = s$.

This way, by means of the transformed complex integral and the definition of the H-function in (3), (12) can be rewritten as:

$$f_Z(x;\overline{\sigma},\overline{\mu},\overline{b}) = H^{N,N}_{N,N}\left[e^{x-\sum_{i=1}^{N} b_i\mu_i}\ \middle|\ \begin{matrix}(0,\sigma_1|b_1|),...,(0,\sigma_N|b_N|)\\(1,\sigma_1|b_1|),...,(1,\sigma_N|b_N|)\end{matrix}\right]. \tag{13}$$

Equation (13) provides a closed form exact representation for the probability density function of the linear combination of logistic random variables. It is worth noticing that (13) is valid for every parameter $\sigma_i > 0$, $\mu_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$, $i = 1,...,N$.

Since the representation is given in terms of a well-known function, it is easy to obtain the cumulative distribution function, as shown in the next subsection.

3.4 The Cumulative Distribution Function of the Linear Combination of Logistic Random Variables

By definition, the cumulative distribution function of the random variable Z, F_Z, whose p.d.f. is given in (13), is obtained as:

$$F_Z(x;\overline{\sigma},\overline{\mu},\overline{b}) = \int_{-\infty}^{x} H^{N,N}_{N,N}\left[e^{x-\sum_{i=1}^{N} b_i\mu_i}\ \middle|\ \begin{matrix}(0,\sigma_1|b_1|),...,(0,\sigma_N|b_N|)\\(1,\sigma_1|b_1|),...,(1,\sigma_N|b_N|)\end{matrix}\right] dx. \tag{14}$$

By considering the definition in (3), a well-known property of the H function provides (Mathai et al., 2010):

$$F_Z(x;\overline{\sigma},\overline{\mu},\overline{b}) = H^{N,N+1}_{N+1,N+1}\left[e^{x-\sum_{i=1}^{N} b_i\mu_i}\ \middle|\ \begin{matrix}(0,\sigma_1|b_1|),...,(0,\sigma_N|b_N|),(1,1)\\(1,\sigma_1|b_1|),...,(1,\sigma_N|b_N|),(0,1)\end{matrix}\right]. \tag{15}$$

It is worth noticing that as (13), (15) is valid for every set of parameters. Being the general analytical representation for the p.d.f. and c.d.f. of the linear combination of logistic random variables given, one shall proceed to obtain the reliability analysis enabled by the latter.

4. Reliability $P(X < Y)$

As discussed in the Introduction, the reliability measure $R = P(X < Y) = P(X - Y < 0)$ is of great interest to both pure and applied scientists. It can be seen that by means of (15), R is easily given by considering the difference of two Logistic random variables.

In the present section, the exact value of R is provided in terms of the H-function, as shall be seen subsequently. Also, an approximation is built based on curve fitting methods in order to facilitate the usage of the formulas hereby developed.

4.1 Exact Expression

Let $X \sim L(\mu_1, \sigma_1)$ and $Y \sim L(\mu_2, \sigma_2)$. Then, by means of (15), $R = P(X < Y) = P(X - Y < 0)$ can be exactly given as:

$$R = H_{3,3}^{2,3}\left[e^{\mu_2-\mu_1}\,\middle|\,\begin{matrix}(0,\sigma_1),(0,\sigma_2),(1,1)\\(1,\sigma_1),(1,\sigma_2),(0,1)\end{matrix}\right]. \tag{16}$$

Even though R is exactly expressed in (16), a mathematical software such as Mathematica must be available for its evaluation. In fact, a Mathematica routine is used to evaluate the H-function, as shown subsequently in the applications section. On the other hand, when out-of-computer quick calculations are needed, a simpler expression in terms of elementary functions is of great interest. In the next subsection, a highly accurate approximation is derived for R.

4.2 Approximated Expression

Curve fitting methods have shown to be important tools to applied scientists. In the present subsection, an approximation to (16) is derived.

The first step in a curve fitting procedure is to consider the number of variables on which the fitted function will depend. In our case, the fitted function is R and, in a first look to (16), the H-function depends on three variables v: $v_1 = \mu_2 - \mu_2$, $v_2 = \sigma_1$ and $v_3 = \sigma_2$. This number can be reduced to 2 by means of the following property of the H-function (Mathai et al., 2010):

$$H_{p,q}^{m,n}\left[z|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right] = kH_{p,q}^{m,n}\left[z^k|\begin{matrix}(a_p, kA_p)\\(b_q, kB_q)\end{matrix}\right], \quad k > 0. \tag{17}$$

Thus, by means of (16) and (17), R can be given as:

$$R = \frac{1}{\sigma_1}H_{3,3}^{2,3}\left[e^{\frac{\mu_2-\mu_1}{\sigma_1}}\,\middle|\,\begin{matrix}(0,\frac{\sigma_2}{\sigma_1}),(0,1),(1,\frac{1}{\sigma_1})\\(1,\frac{\sigma_2}{\sigma_1}),(1,1),(0,\frac{1}{\sigma_1})\end{matrix}\right]. \tag{18}$$

Equation (18) may be further simplified by looking at its integral representation given in (3). Thus, (18) may be rewritten as:

$$R = H_{3,3}^{2,3}\left[e^{\frac{\mu_2-\mu_1}{\sigma_1}}\,\middle|\,\begin{matrix}(0,\frac{\sigma_2}{\sigma_1}),(0,1),(1,1)\\(1,\frac{\sigma_2}{\sigma_1}),(1,1),(0,1)\end{matrix}\right]. \tag{19}$$

By looking at (19), the H-function now only depends on two variables v: $v_1 = e^{\frac{\mu_1-\mu_2}{\sigma_1}}$ and $v_2 = \frac{\sigma_2}{\sigma_1}$. It is now of interest to study the range of variation of the variables v_1 and v_2.

The variable v_1 satisfies the inequalities:

$$\begin{array}{c} 0 < v_1 \leq 1, \quad \text{for} \quad \mu_1 \leq \mu_2 \\ \text{and} \\ v_1 > 1, \quad \text{for} \quad \mu_1 > \mu_2 \end{array} \tag{20}$$

In a curve fitting procedure, it is interesting to sample the variables in their whole domain. In the case under consideration, both v_1 and v_2 can be positive real numbers, thus it is impossible to have a good density of sampling points. This can be overcome by considering only the case in which $\mu_1 \leq \mu_2$ because the complement can be given by using the relation $P(X < Y) = 1 - P(Y < X)$. In other words, the expression to be derived is fitted for the case where $\mu_1 \leq \mu_2$, such that when the opposite occurs, the fitted formula will still apply to evaluate $P(Y < X)$, and so $P(X < Y)$ by the relation above described.

By the considerations above, the range of the variables to be considered in the curve fitting are $0 \leq v_1 \leq 1$ and $0 \leq v_2 < \infty$.

The approximating function whose coefficients d and r are to be determined is:

$$(1-d)\left(\frac{1-2d}{2-2d}\right)^{v_1^r} + d \tag{21}$$

in which $0 < d < 1/2$ and $r > 0$ are both functions of v_2.

It is interesting to notice that (21) obeys the conditions:

- When $\mu_1 = \mu_2$, $v_1 = 1$ and $R = 1/2$;
- When $\mu_2 >>> \mu_1$, $v_1 = 0$ and $R = 1$.

In order to obtain the fitted function, the following procedure has been carried out:

- At first, a value w, $0 < w < 10$ has been sampled. By taking $v_2 = w$, the H-function in (18) has been sampled for $v_1 = 0.01k$, $k = 0, ..., 100$;
- For each set of data obtained from the last step, the nonlinear fit tool of the software Mathematica has been used to determine the best parameters d and r according to (21);
- The procedure of the last two steps has been repeated 100 times, obtaining the values of v_2 and the corresponding values of d and r;

- The behavior of both d and r has been studied and they were fitted as other functions of v_2.

By following the procedure above, d and r are given as:

$$d(v_2) = 0.208 + 0.757e^{-\frac{2.020}{0.670+v_2}}(0.670 + v_2)^{-3/2}, \tag{22}$$

and

$$r(v_2) = 0.023 + 2.410e^{-\frac{1.044}{0.591+v_2}}(0.591 + v_2)^{-3/2}. \tag{23}$$

This way, the approximation to R is given by means of (22) and (23) as:

$$R = \begin{cases} (1-d)\left(\frac{1-2d}{2-2d}\right)^{v_1^r} + d, \quad v_1 = e^{\frac{\mu_1-\mu_2}{\sigma_1}}, \quad v_2 = \frac{\sigma_2}{\sigma_1}; \quad \mu_1 \leq \mu_2 \\ 1 - d - (1-d)\left(\frac{1-2d}{2-2d}\right)^{v_1^r}, \quad v_1 = e^{\frac{\mu_2-\mu_1}{\sigma_2}}, \quad v_2 = \frac{\sigma_1}{\sigma_2}; \quad \mu_1 > \mu_2 \end{cases} . \tag{24}$$

The maximum absolute error between (16) and (24) is less than 0.008 within the fitting interval, indicating a very accurate approximation.

In the next section a few numerical applications of the results hereby obtained are shown.

5. Numerical Applications of the Results: Reliability of Logistic Distributions

In this section, the formulas developed in the present paper are numerically evaluated in order to show their applicability.

5.1 Reliability of the type $R = P(X < Y)$

When only two logistic distributions are considered, the present paper provides both the exact and approximated formulas for evaluating the reliability measure R. In order to show the applicability of (16) and (24), a set of logistic random variables is shown in Table 1 and graphically in Figure 1. It is worth noticing that the representations which involve the H-function have been numerically evaluated by means of a computational code in Mathematica.

Table 1: Logistic Random Variables Considered $R = P(X < Y)$

Random Variable	μ	σ
X_1	-1.5	2
X_2	-2.5	3.25
X_3	1.75	2.75
X_4	3.25	1.25

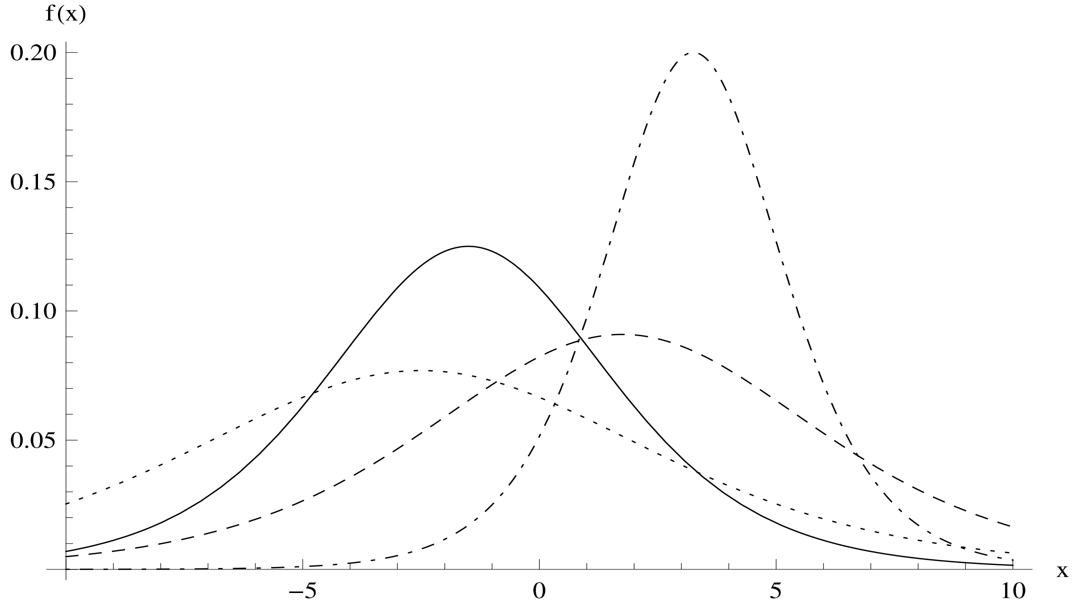

Figure 1: Probability Density Functions of the Random Variables from Table 1 (X_1 full, X_2 dotted, X_3 dashed and X_4 dot-dashed).

By means of the logistic distributions considered in Table 1, the reliability measures in Table 2 can be obtained:

Table 2: Reliability Measures $R = P(X < Y)$

Reliability Measure	From (16)	From (24)	Estimated From Random Data	Variance of Estimator
$P(X_1 < X_2)$	0.43793	0.43984	0.43790	2.5126×10^{-6}
$P(X_2 < X_3)$	0.72093	0.72045	0.72090	1.9199×10^{-6}
$P(X_3 < X_4)$	0.61753	0.61646	0.61757	2.3772×10^{-6}

The values of R estimated from data have been obtained by following the procedure below:

- Generate random samples with 10^5 elements each for the random variables X and Y;

- Let x_i and y_i, $i = 1, ..., 10^5$, denote the elements of the random samples of the random variables X and Y, respectively. Consider the indicator function

$I(x, y) = 1 - u(x - y)$, where $u(x) = 0$, $x < 0$ and $u(x) = 1$, otherwise. The value of R can be estimated as $R_e = \sum_{i=1}^{10^5} I(x_i, y_i)$;

- Repeat the above process 1000 times and then take the mean value of the R_es generated. This value is shown in Table 2. The variance of the R_es generated is also shown in Table 2.

5.1.1 Previous Results

Haghighi and Shayib (2010) studied the reliability R for independent random variables X and Y following logistic distributions with null location parameters. The authors presented a series expression for R dependent on the scale parameters of both distributions being compared.

Fundamentally the result in (Haghighi and Shayib (2010)) is wrong as $R = 1/2$ whenever the distributions being compared are of the same type, symmetrical and with same mean value. In the present paper, the correct results are obtained.

5.2 Reliability of the type $R = P(X + Y < Z + W)$

In the case in which the sum of two logistic random variables is compared to another sum of different logistic random variables, the reliability measure $R = P(X+Y < Z+W)$ can be easily obtained from (15). A few numerical simulations with the random variables from Table 1 are shown in Table 3.

Table 3: Reliability Measures $R = P(X + Y < Z + W)$

Reliability Measure	From (15)	Estimated From Random Data	Variance of Estimator
$P(X_1 + X_2 < X_3 + X_4)$	0.85304	0.85300	1.3229×10^{-6}
$P(X_1 + X_3 < X_2 + X_4)$	0.52364	0.52369	2.4787×10^{-6}
$P(X_1 + X_4 < X_2 + X_3)$	0.38357	0.38359	2.3592×10^{-6}

The estimated values in Table 3 are evaluated similarly to the ones of Table 2. The only difference is that before applying the indicator function, the random samples of the variables X and Y are element-wise summed up. The same is done to the random samples of the variables Z and W.

Table 3 reveals a very good accordance between the H-function value of the reliability measure and the estimated value, which corroborates to the validity of the equations hereby proposed.

5.3 Reliability of the type $R = P(X + Y + Z < W + P + Q)$

Finally, when the sum of three logistic random variables is to be compared to the sum of different three logistic random variables, the result (15) easily provides the exact values of the reliability measures involved.

Table 4 presents a few numerical simulations by considering the random variables from Table 1.

Table 4: Reliability Measures $R = P(X + Y + Z < W + P + Q)$

Reliability Measure	From (15)	Estimated From Rnd. Data	Variance of Estimator
$P(X_1 + X_2 + X_3 < X_1 + X_2 + X_4)$	0.55482	0.55483	2.3285×10^{-6}
$P(X_1 + X_2 + X_4 < X_2 + X_3 + X_4)$	0.62655	0.62145	2.2754×10^{-6}
$P(X_1 + X_3 + X_4 < X_1 + X_2 + X_3)$	0.29170	0.29174	2.1564×10^{-6}

As in the case of the previous subsection, Table 4 shows that the results are well predicted by the analytical formulas hereby developed.

The estimated values were obtained by a similar procedure as the one considered to build Table 3. The difference is that in the former, each set of three random samples has been summed up element-wise before applying the indicator function.

6. Conclusions

Logistic random variables have shown to be a simple yet powerful model for variables concerning both pure and applied scientists. In general, while dealing with random variables, obtaining the distribution of the linear combination of the latter is of interest.

In the present paper, the probability density function and the cumulative distribution function of the linear combination of N independent and not identically distributed logistic random variables has been obtained in terms of the H-function. These expressions have been used to derive reliability measures of the type $P(\sum_{i=1}^{N_1} X_i < \sum_{j=1}^{N_2} Y_j)$ when X_i, $i = 1, ..., N_1$ and Y_j, $j = 1, ..., N_2$ are logistic random variables. A highly accurate approximate expression has been built for the case $N_1 = N_2 = 1$ by means of curve fitting techniques.

The applicability of the expressions developed has been verified by numerical experiments, revealing a very good accordance between the exact and the estimated reliability measures.

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