J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.5, No.1 (2016), pp. 147-154

ON INTEGRAL OPERATOR INVOLVING MITTAG-LEFFLER FUNCTION

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Dedicated to Prof. M.A. Pathan on his 75^{th} birth anniversary

Abstract: The main object of this article is to present an interesting double integral involving generalized Mittag-Leffler function, which is expressed in terms of generalized (Wright) hypergeometric function. Also we consider some special cases as an application of main result.

Keywords: Mittag-Leffler function, Generalized (Wright) hypergeometric function and Integrals.

2000 AMS Subject Classifications: 33C45, 33C60, 33E12.

1. Introduction and Definition

The well known Mittag-Leffler function is defined as follows (see, [4]):

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \tag{1.1}$$

where $\alpha \in C,\ \Re(\alpha) > 0, z \in C$ and its general form is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}, \tag{1.2}$$

where $\alpha, \beta \in C, \Re(\alpha) > 0, \Re(\beta) > 0, z \in C$ with C being the set of complex numbers which is known as Wiman function [10].

In 1971, Prabhakar [5] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ as follows:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)\ n!}, \tag{1.3}$$

where $\alpha, \beta, \gamma \in C$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, z \in C$ and $(\gamma)_n$ is the well known Pochhammer symbol [6].

In continuation of his work, Shukla and Prajapati [8] introduced the following extension of Mittag-Leffler function:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)\ n!}, \tag{1.4}$$

where $\alpha, \beta, \gamma \in C$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \bigcup N$.

The function $E^{\gamma,q}_{\alpha,\beta}(z)$ is most natural generalization of the exponential function $\exp(z)$, Mittag-Leffler function $E_{\alpha}(z)$ and Wiman function $E_{\alpha,\beta}(z)$. Furthermore, the function $E^{\gamma,q}_{\alpha,\beta}(z)$ has the following special cases (see, [3]):

$$E^{\gamma,1}_{\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z), E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), E_{\alpha,1}(z) = E_{\alpha}(z),$$

$$E_{1,2}(z) = \frac{e^z - 1}{z}, E^{1,1}_{1,1}(z) = E_{1,1}(z) = E_1(z) = \exp(z),\ z \in C. \tag{1.5}$$

The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [2] and Wright ([11], [12], [13]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [9, p.21]; see also [7]):

$${}_p\Psi_q\left[\begin{matrix} (\alpha_1\ ,\ A_1), & \ldots.., & (\alpha_p\ ,\ A_p); \\ & & \\ (\beta_1\ ,\ B_1), & \ldots.., & (\beta_q\ ,\ B_q); \end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!} \tag{1.6}$$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$\text{(i)}\ \ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0 \text{ and } 0 < |z| < \infty;\ z \neq 0. \tag{1.6a}$$

$$\text{(ii)}\ \ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j = 0 \text{ and } 0 < |z| < {A_1}^{-A_1} \ldots {A_p}^{-A_p} {B_1}^{B_1} \ldots {B_q}^{B_q}. \tag{1.6b}$$

A special case of (1.6) is

$$
{}_p\Psi_q\left[\begin{matrix}(\alpha_1\ ,\ 1), & \ldots.., & (\alpha_p\ ,\ 1); \\ & & & z \\ (\beta_1\ ,\ 1), & \ldots.., & (\beta_q\ ,\ 1); \end{matrix}\right] = \frac{\prod\limits_{j=1}^{p}\ \Gamma(\alpha_j)}{\prod\limits_{j=1}^{q}\ \Gamma(\beta_j)}\ {}_pF_q\left[\begin{matrix}\alpha_1, & \ldots.., & \alpha_p\ ; \\ & & & z \\ \beta_1, & \ldots.., & \beta_q\ ; \end{matrix}\right], \tag{1.7}
$$

where ${}_pF_q$ is the generalized hypergeometric series defined by [6]

$$
{}_pF_q\left[\begin{matrix}\alpha_1, & \ldots.., & \alpha_p\ ; \\ & & & z \\ \beta_1, & \ldots.., & \beta_q\ ; \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\ \frac{z^n}{n!}
$$

$$
= {}_pF_q(\alpha_1,\cdots,\alpha_p;\ \beta_1,\cdots\beta_q;\ z), \tag{1.8}
$$

where $(\lambda)_n$ is the Pochhammer's symbol [6].

2. Useful Result

For our present investigation, the following interesting and useful result due to Edward [1, p.445] will be required:

$$
\int_0^1\int_0^1 y^{\alpha}\ (1-x)^{\alpha-1}\ (1-y)^{\beta-1}\ (1-xy)^{1-\alpha-\beta}\ dx\ dy = \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{2.1}
$$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

3. Main Result

We establish here the following double integral involving Mittag-Leffler function $E^{\gamma,\nu}_{\alpha,\beta}$, which is expressed in terms of Wright hypergeometric function:

$$
\int_0^1\int_0^1 y^{\lambda}\ (1-x)^{\lambda-1}\ (1-y)^{\mu-1}\ (1-xy)^{1-\lambda-\mu}\ E^{\gamma,\nu}_{\alpha,\beta}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\ dx\ dy
$$

$$
= \frac{1}{\Gamma(\gamma)}\ {}_3\Psi_2\left[\begin{matrix}(\gamma,\ \nu), & (\lambda,\ 1), & (\mu,\ 1)\ ; & \\ & & & a \\ (\beta\ ,\ \alpha), & (\lambda+\mu,\ 2) & ; & \end{matrix}\right], \tag{3.1}
$$

where a is nonzero constant, $\Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(\gamma) > 0,\ \Re(\lambda) > 0,\ \Re(\mu) > 0$, $E^{\gamma,\nu}_{\alpha,\beta}$ and ${}_3\Psi_2$ are the Mittag-Leffler and Wright hypergeometric functions defined by (1.4) and (1.5), respectively.

Proof : To establish our main result (3.1), we denote the left-hand side of (3.1) by I and then using (1.4), we have

$$I = \int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu} \sum_{n=0}^{\infty} \frac{(\gamma)_{\nu n}\,\left\{\frac{ay(1-x)(1-y)}{(1-xy)^2}\right\}^n}{\Gamma(\alpha n+\beta)\,n!}\,dx\,dy. \tag{3.2}$$

Now changing the order of integration and summation, which is clearly seen to be justified due to the uniform convergence of the series in the interval (0,1), we arrive at

$$I = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+\nu n)\,\Gamma(\lambda+n)\,\Gamma(\mu+n)\,(a)^n}{\Gamma(\alpha n+\beta)\,\Gamma(\lambda+\mu+2n)\,n!}. \tag{3.3}$$

Finally, summing up the above series with the help of (1.5), we easily arrive at the right-hand side of (3.1). This completes the proof of our main result.

Next, we consider other variation of (3.1). In fact, we establish an integral formula for the Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\nu}$, which is expressed in terms of the generalized hypergeometric function ${}_pF_q$.

4. Variation of (3.1): Let the conditions of our main result be satisfied, then the following integral formula holds true:

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E_{\alpha,\beta}^{\gamma,\nu}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\beta)\,\Gamma(\lambda+\mu)}\;{}_{\nu+2}F_{\alpha+2}\left[\begin{matrix}\Delta(\nu;\ \gamma), & \lambda, & \mu\ ; \\ \Delta(\alpha;\ \beta), & \Delta(2;\ \lambda+\mu) & ;\end{matrix}\ \frac{a\nu^{\nu}}{4\alpha^{\alpha}}\right], \tag{4.1}$$

where $\Delta(m;\ l)$ abbreviates the array of m parameters $\frac{l}{m},\ \frac{l+1}{m},\ \cdots,\ \frac{l+m-1}{m}$, $m \geq 1$.

Proof: In order to prove the result (4.1), using the results

$$\Gamma(\alpha+n) = \Gamma(\alpha)(\alpha)_n$$

and

$$(l)_{kn} = k^{kn}\left(\frac{l}{k}\right)_n \left(\frac{l+1}{k}\right)_n \cdots \left(\frac{l+k-1}{k}\right)_n.$$

(Gauss multiplication theorem) in (3.3) and summing up the given series with the help of (1.8), we easily arrive at our required result (4.1).

5. Special Cases :

(i). On taking $\nu = 1$ in (3.1) and by using $E^{\gamma,\nu}_{\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z)$, we get

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E^{\gamma}_{\alpha,\beta}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\,dx\,dy$$

$$= \frac{1}{\Gamma(\gamma)}\,{}_3\Psi_2\left[\begin{array}{lll} (\gamma,\ 1), & (\lambda,\ 1), & (\mu,\ 1)\ ; \\ & & \qquad a \\ (\beta\ ,\ \alpha), & (\lambda+\mu,\ 2) & \qquad ; \end{array}\right], \tag{5.1}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ and $E^{\gamma}_{\alpha,\beta}$ is the Mittag-Leffler function defined by (1.3).

(ii). On setting $\gamma = 1$ in (5.1) and by using $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$, we get

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E_{\alpha,\beta}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\,dx\,dy$$

$$= {}_3\Psi_2\left[\begin{array}{lll} (1,\ 1), & (\lambda,\ 1), & (\mu,\ 1)\ ; \\ & & \qquad a \\ (\beta\ ,\ \alpha), & (\lambda+\mu,\ 2) & \qquad ; \end{array}\right], \tag{5.2}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ and $E_{\alpha,\beta}$ is the Mittag-Leffler function defined by (1.2).

(iii). On putting $\beta\ = 1$ in (5.2) and by using $E_{\alpha,1}(z) = E_{\alpha}(z)$, we get

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,E_{\alpha}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\,dx\,dy$$

$$= {}_3\Psi_2\left[\begin{array}{lll} (1,\ 1), & (\lambda,\ 1), & (\mu,\ 1)\ ; \\ & & \qquad a \\ (1\ ,\ \alpha), & (\lambda+\mu,\ 2) & \qquad ; \end{array}\right], \tag{5.3}$$

where $\Re(\alpha) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ and E_{α} is the Mittag-Leffler function defined by (1.1).

(iv). On taking $\alpha\ = 1$ in (5.3) and by using $E_1(z) = \exp(z)$, we get

$$\int_0^1 \int_0^1 y^{\lambda}\,(1-x)^{\lambda-1}\,(1-y)^{\mu-1}\,(1-xy)^{1-\lambda-\mu}\,\exp\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\,dx\,dy$$

$$= {}_2\Psi_1\left[\begin{matrix}(\lambda,\ 1), & (\mu,\ 1) & ; \\ & & & a \\ (\lambda+\nu\ ,\ 2) & & ;\end{matrix}\right], \tag{5.4}$$

where $\Re(\gamma) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$.

(v). On setting $\alpha\ = 1,\ \beta\ = 2$ in (5.2) and by using $E_{1,2}(z) = \frac{e^z-1}{z}$, we get

$$\int_0^1\int_0^1 y^{\lambda-1}(1-x)^{\lambda-2}(1-y)^{\mu-2}(1-xy)^{3-\lambda-\mu}\left[\exp\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]-1\right]dxdy$$

$$= {}_3\Psi_2\left[\begin{matrix}(1,\ 1), & (\lambda,\ 1), & (\mu,\ 1) & ; \\ & & & & a \\ (2\ ,\ 1), & (\lambda+\mu,\ 2) & & ;\end{matrix}\right], \tag{5.5}$$

where $\Re(\mu) > 0, \Re(\lambda) > 0$.

(vi). On taking $\nu = 1$ in (4.1), we get

$$\int_0^1\int_0^1 y^{\lambda}\ (1-x)^{\lambda-1}\ (1-y)^{\mu-1}\ (1-xy)^{1-\lambda-\mu}\ E^{\gamma}_{\alpha,\beta}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\ dx\ dy$$

$$= \frac{\Gamma(\lambda)\ \Gamma(\mu)}{\Gamma(\beta)\ \Gamma(\lambda+\mu)}\ {}_3F_{\alpha+2}\left[\begin{matrix}\gamma & \lambda & \mu\ ; \\ & & & \frac{a}{4\alpha^{\alpha}} \\ \Delta(\alpha;\ \beta), & \Delta(2;\ \lambda+\mu); & \end{matrix}\right], \tag{5.6}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$.

(vii). On putting $\gamma =\ 1$ in (5.6), we get

$$\int_0^1\int_0^1 y^{\lambda}\ (1-x)^{\lambda-1}\ (1-y)^{\mu-1}\ (1-xy)^{1-\lambda-\mu}\ E_{\alpha,\beta}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\ dx\ dy$$

$$= \frac{\Gamma(\lambda)\ \Gamma(\mu)}{\Gamma(\beta)\ \Gamma(\lambda+\mu)}\ {}_3F_{\alpha+2}\left[\begin{matrix}1 & \lambda & \mu\ ; \\ & & & \frac{a}{4\alpha^{\alpha}} \\ \Delta(\alpha;\ \beta), & \Delta(2;\ \lambda+\mu); & \end{matrix}\right], \tag{5.7}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$.

(viii). Further, on setting $\beta =\ 1$ in (5.7), we get

$$\int_0^1\int_0^1 y^{\lambda}\ (1-x)^{\lambda-1}\ (1-y)^{\mu-1}\ (1-xy)^{1-\lambda-\mu}\ E_{\alpha}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\ dx\ dy$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)}\ {}_3F_{\alpha+2}\left[\begin{matrix} 1 & \lambda & \mu\ ; \\ \Delta(\alpha;\ 1), & \Delta(2;\ \lambda+\mu); & \end{matrix}\ \frac{a}{4\alpha^{\alpha}}\right], \tag{5.8}$$

where $\Re(\alpha) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$.

(ix). On taking $\alpha\ = 1$ in (5.8), we get

$$\int_0^1\int_0^1 y^{\lambda}\ (1-x)^{\lambda-1}\ (1-y)^{\mu-1}\ (1-xy)^{1-\lambda-\mu}\ \exp\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]\ dx\ dy$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)}\ {}_2F_2\left[\begin{matrix} \lambda & \mu\ ; \\ \Delta(2;\ \lambda+\mu); & \end{matrix}\ \frac{a}{4}\right], \tag{5.9}$$

where $\Re(\mu) > 0, \Re(\lambda) > 0$.

(x). On putting $\alpha\ = 1,\ \beta\ = 2$ in (5.7), we get

$$\int_0^1\int_0^1 y^{\lambda-1}(1-x)^{\lambda-2}(1-y)^{\mu-2}(1-xy)^{3-\lambda-\mu}\left[\exp\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]-1\right]dxdy$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)}\ {}_3F_3\left[\begin{matrix} 1 & \lambda & \mu\ ; \\ 2 & \Delta(2;\ \lambda+\mu); & \end{matrix}\ \frac{a}{4}\right], \tag{5.10}$$

where $\Re(\mu) > 0, \Re(\lambda) > 0$.

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