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ULTRA GAMMA FUNCTION, PROPERTIES AND APPLICATIONS: A PRODIGY

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Dedicated to Prof. M.A. Pathan on his 75^{th} birth anniversary

Abstract: In the present paper we consider a unified integral transform *"Ultra Gamma Transform"* (UGT) and its relation with fractional integral operators. We list out certain properties of this transform and present a table of UGT of elementary functions including trigonometric function as well as hyperbolic function in terms of generalized three parameter gamma function renamed as *"Ultra Gamma Function"* (UGF). We also point out certain special cases of this integral transform in terms of well-known classical integral transforms, other integral transforms and generalized gamma function that occurs in the study of scattering of waves and that with the classical gamma function in support of justification of the findings. Finally some statistical affiliations of UGF are reported from real life applications point of view.

Keywords: Gamma Functions, Integral Transform, Fractional Integral Operators, Probability Distribution.

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1. Introduction

The classical gamma function is the key function in the development of special functions of importance in various Scientific and Engineering problems (See Erdélyi [7]). Kobayashi [13] formally defined the generalization of classical gamma function, which appears in the Weiner-Hopf technique (See Noble [18]) of dealing with the problem of wave scattering. Kobayashi's generalized function involves two parameters and readily yields the classical gamma function as a special case.

Motivated by the work of Kobayashi; Banerji and Sinha [5] considered a three parameter generalization of Kobayashi's gamma function and studied its asymptotic and characteristic properties. Wonju [30] studied the theory of diffraction where he emphasized on analytical treatment of Kobayashi's generalized gamma function and gives the exact evaluations of the same. The study of such functions have attracted many mathematicians who have generalized and studied this function. To name we include Al-Musallam and Kalla [1], [2], Galu*é* et al. [8], Saxena and Kalla [22], Srivastava et al. [23] and Virchenko et al. [24]. Many authors, such as Al-Zamel [3], Ali et al. [4], Kalla and Saqabi [10], Kalla et al. [11] have considered the statistical applications of generalized gamma-type function. El-Fateh et al. [6] have provided the practical applications of modified gamma function in mathematical modeling of inventory control problems.

The aim of this paper is to introduce and study a unified integral transform (UGT) consisting of three parameters and prepare a table of this transform of well known elementary functions in terms of the ultra gamma function (UGF). Paper is organized in five sections. In section 1 we provide the interpretation of UGF $\Gamma_\lambda(u, v, s)$, earlier considered by Banerji and Sinha [5], in terms of Fox's H-function and some important properties of it. Section 2 deals with ultra gamma transform, the theorem of existence, relation with classical integral transforms and its properties. Section 3 exhibits the relation of UGT with Weyl's fractional integral operator and representation of UGF as a fractional integral of product of elementary functions. The solution of integral equation describing the Wiener-Hopf technique is presented using Weyl fractional integral operator. Section 4 is devoted to the evaluations of UGT of certain elementary functions, trigonometric functions, hyperbolic functions and the combinations thereof. The concluding section embodies the probability density associated with UGF and its useful statistical affiliations such as moments, moment generating function, cumulative density, hazard function, survivor function and the mean residue life function, which find applications in vital statistics or demography. Extensive bibliography provides the collection of study of gamma-type functions and their generalizations at a glance. Appendix, at the end, embodies the definition of Fox's H-function and its properties that are used in the findings of this paper. It also contains the glimpses of integral equation and its solution in terms of Kobayashi's generalized gamma function, which describes the process of Weiner-Hopf technique.

1.1 Ultra Gamma Function (UGF)

Let λ be a positive integer, $\Re(u) > 0, \Re(s) > 0, |v| > 0, |\arg v| < \pi$ and t^{u-1} be interpreted as a principal value. Then three parameter generalized gamma function renamed as *ultra gamma function*, earlier considered by Banerji and Sinha [5], is defined by

$$\Gamma_\lambda(u,v,s)=\int_0^\infty \frac{t^{u-1}e^{-st}}{(t+v)^\lambda}\,dt, \tag{1}$$

which has the convergence under the conditions stated herewith. Banerji and Sinha [5] further provided the analysis of generalized psi-function denoted by $\psi_\lambda(u,v,s)$, incomplete generalized gamma function denoted by $\Gamma_\lambda(u,v,s,w)$ and the partial differential coefficients of UGF with respect to the parameters involved.

1.1.1 Interpretation of UGF in terms of Fox's H-Fnction

The UGF defined in equation (1), with a couple of mathematical manipulations, can be represented in terms of Fox's H-function. A detailed account of H-function is available in Mathai and Saxena [14]. Following theorem establishes the relation of UGF with H-function:

Theorem 1: Let the conditions stated with the definition of UGF be satisfied. Then the UGF possesses following H-function representations:

$$\Gamma_\lambda(u,v,s)=\frac{s^{\lambda-1}}{\Gamma(\lambda)}H^{1,2}_{1,1}\left[\begin{array}{c}(u,1),(\lambda,1)\\(u+\lambda-1)\end{array}\middle|\frac{1}{sv}\right], \tag{2}$$

which can also be expressed as

$$\Gamma_\lambda(u,v,s)=\frac{s^{\lambda-1}}{\Gamma(\lambda)}H^{2,1}_{1,1}\left[\begin{array}{c}(2-u-\lambda,1)\\(1-u,1),(1-\lambda,1)\end{array}\middle|sv\right]. \tag{3}$$

Proof: Rewrite the equation (1) as follows:

$$\Gamma_\lambda(u,v,s)=\int_0^\infty t^{u-1}e^{-st}(t+v)^{-\lambda}\,dt, \tag{4}$$

the right hand side of which can further be expressed as

$$=v^{-\lambda}\int_0^\infty t^{u-1}e^{-st}\left[1+\frac{t}{v}\right]^{-\lambda}\,dt.$$

Invoking the H-function representation of a binomial series [See Mathai and Saxena [14], p. 10; eq. (1.7.3)], we obtain

$$=\frac{v^{-\lambda}}{\Gamma(\lambda)}\int_0^\infty t^{u-1}e^{-st}\;H^{1,1}_{1,1}\left[\begin{array}{c}(1-\lambda,1)\\(1,1)\end{array}\middle|\frac{t}{v}\right]\,dt.$$

Using the property of H-function [See Mathai and Saxena [14], p.4, eq. (1.2.4) also provided in the appendix at the end of this paper], we have

$$= \frac{v^{-\lambda}}{\Gamma(\lambda)} \int_0^\infty e^{-st} \ H_{1,1}^{1,1} \left[\begin{array}{c} (u-\lambda, 1) \\ (u-1, 1) \end{array} \middle| \frac{t}{v} \right] \ dt.$$

Owing to the definition of Laplace transform of H-function, we finally obtain

$$= \frac{v^{-\lambda}}{s\Gamma(\lambda)} H_{1,1}^{1,2} \left[\begin{array}{c} (u-\lambda, 1), (0, 1) \\ (u-1, 1) \end{array} \middle| \frac{1}{sv} \right],$$

which completes the proof of the theorem with the aid of property of H-function (Mathai and Saxena [14], p.4, equation (1.2.4)). The use of Mathai and Saxena [14], p.4, equation (1.2.2), the alternative representation given in (3) is justified.
Note: Taking $s = 1$ and $\lambda = m$ in the results of this theorem, Kobayashi's generalized gamma function can be interpreted in terms of Fox's H-function as a special case.

$$\Gamma_m(u, v, 1) = \Gamma_m(u, v) = \frac{1}{\Gamma(m)} H_{1,1}^{1,2} \left[\begin{array}{c} (u, 1), (m, 1) \\ (u+m-1, 1) \end{array} \middle| \frac{1}{v} \right]$$

$$= \frac{1}{\Gamma(m)} H_{1,1}^{2,1} \left[\begin{array}{c} (2-u-m, 1) \\ (1-u, 1), (1-m, 1) \end{array} \middle| v \right]$$

1.1.2 Asymptotic Property of UGF

Asymptotic expansions are helpful in determining the nature of functions for higher values of the variable for which they are defined. We reproduce the important asymptotic property of UGF given by equation (1) (See Banerji and Sinha [5]), which is otherwise very important from the point of view of existence of this transform. The UGF $\Gamma_\lambda(u, v, s)$ has an asymptotic expansion of the form

$$\Gamma_\lambda(u, v, s) \sim \sum_{n=0}^{\infty} C(-\lambda, n) \frac{\Gamma(u+n)}{v^{n+\lambda} s^{u+n}}, \tag{5}$$

uniformly in generalized sense in arg$v$ as $v \to \infty$ and $s \to \infty$ for a well defined domain of v and s. Also, $C(a, b)$ stands for binomial coefficients.

1.1.3 Recurrence Relations of UGF

It is justified to reproduce the recurrence relations due to Banerji and Sinha [5], which will be helpful in the further findings of this paper.

$$\Gamma_\lambda(u+m,v,s) = \frac{u+m-1}{s}\,\Gamma_\lambda(u+m-1,v,s) - \frac{\lambda}{s}\,\Gamma_{\lambda+1}(u+m,v,s); \tag{6}$$

$$\Gamma_{\lambda+r}(u,v,s) = \frac{u-1}{\lambda+r-1}\,\Gamma_{\lambda+r-1}(u-1,v,s) - \frac{s}{\lambda+r-1}\,\Gamma_{\lambda+r-1}(u,v,s); \tag{7}$$

$$\Gamma_{\lambda+m}(u+m,v,s) = \frac{u+m-1}{\lambda+m-1}\Gamma_{\lambda+m-1}(u+m-1,v,s)$$

$$-\frac{s}{\lambda+m-1}\Gamma_{\lambda+m-1}(u+m,v,s). \tag{8}$$

1.2 Particular Cases of UGF

It is worth noting the particular cases of ultra gamma function defined in equation (1) as follows:
For $\lambda = m$ and $s = 1$, the gamma function due to Kobayashi is expressed as

$$\Gamma_m(u,v,1) = \Gamma_m(u,v) = \int_0^\infty \frac{t^{u-1}e^{-t}}{(t+v)^m}\,dt, \tag{9}$$

which has great importance in the theory of diffraction related to Weiner-Hopf method of evaluation; because the process of evaluation can explicitly be described by using (9). The generalized gamma function occurs in the form $\Gamma_1(n+1/2, -2i(\kappa+\lambda))$, where n is a non-negative integer as can be seen in the following explicit representation of the solution of integral equation describing the Wiener-Hopf technique:

$$\Psi_+^{1,2}(\lambda) = \mp\sum_{n=0}^{N}\left(\frac{i}{2}\right)^n \frac{d^n\Psi_+^{1,2}(\kappa)}{n!\,d\lambda^n}\sqrt{2}\pi^{-1}(\kappa+\lambda)\times$$

$$e^{i(2\kappa-\pi/4)}\Gamma_1(n+1/2,-2i(\kappa+\lambda)) + S^{1,2}(\lambda) \tag{10}$$

where the symbols have usual meanings and $S^{1,2}(\lambda)$ contains a combination of well-known Fresnel integrals (See Wonju [30]).

Further, for $\lambda = 0, s = 1$, it reduces to classical gamma function widely used in the development of special functions.

$$\Gamma_0(u, v, 1) = \Gamma(u) = \int_0^\infty t^{u-1} e^{-t} \, dt, \tag{11}$$

where v has no impact on this reduction formula.

2. Unified Integral Transform (Ultra Gamma Transform)

Definition 1: Let $f(t), t > 0$ be a bounded measurable function in the interval $(0, \infty)$ and locally integrable in Riemann sense. Also, λ be a positive integer, $\Re(u) > 0, \Re(s) > 0, |v| > 0, |argv| < \pi$ and t^{u-1} be interpreted as a principal value. Then we consider following unified integral transform hereinafter termed as Ultra Gamma Transform (UGT):

$$\Im_{s,\lambda}^{u,v}[f(t)] = \int_0^\infty \frac{t^{u-1} e^{-st}}{(t+v)^\lambda} \, f(t) \, dt \tag{12}$$

Definition 2: If one of the parameters in (12) tends to unity, i.e., $s \to 1$, then the UGT gives rise to an integral transform involving the integrand of (9) as kernel, which is directly responsible for the involvement of Kobayashi's generalized gamma function in further analysis along with the conditions of validity. The integral transform is denoted by $\mathcal{D}_\lambda^{u,v}[f(t)]$ and is expressed as follows:

$$\Im_{1,\lambda}^{u,v}[f(t)] = \mathcal{D}_\lambda^{u,v}[f(t)] = \int_0^\infty \frac{t^{u-1} e^{-t}}{(t+v)^\lambda} \, f(t) \, dt \tag{13}$$

The properties and results obtained for UGT, defined in (12), in the subsequent sections of present paper shall be vis-*á*-vis applicable for $\mathcal{D}_\lambda^{u,v}[f(t)]$ also.

2.1 Existence Theorem

Theorem 2: Let $|f(t)| < M$, for $M > 0$, be a bounded measurable function in the interval $(0, \infty)$ and locally integrable in Riemann sense. Also, λ be a positive integer, $\Re(u) > 0, \Re(s) > 0, |v| > 0, |\arg v| < \pi$ and t^{u-1} be interpreted as a principal value. Then the UGT defined in equation (12) exists for $t > 0$.

Proof: In view of the conditions stated with the theorem and the asymptotic expansion of the UGF given in equation (5), the existence of UGT is justified and hence details are omitted.

2.2 Relation with other Integral Transforms

Specializing the parameters involved in the definition of unified integral transform defined in equation (12), we list out its relation with classical integral transforms and some other integral transforms as given hereunder:

Table 1: Special Cases of Ultra Gamma Transforms

(i)	$\Im_{s,0}^{1,v}[f(t)]$	$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}\ f(t)\ dt$	Laplace Transform
(ii)	$\Im_{0,0}^{u,v}[f(t)]$	$\mathcal{M}[f(t)] = \int_0^\infty t^{u-1}\ f(t)\ dt$	Mellin Transform
(iii)	$\Im_{0,1}^{1,v}[f(t)]$	$\mathcal{S}[f(t)] = \int_0^\infty \frac{f(t)}{(t+v)}\ dt$	Stieltjes Transform
(iv)	$\Im_{0,\lambda}^{1,v}[f(t)]$	$\mathcal{S}_{\text{gen}}[f(t)] = \int_0^\infty \frac{f(t)}{(t+v)^\lambda}\ dt$	Generalized Stieltjes Transform
(v)	$\Im_{s,0}^{u,v}[f(t)]$	$\mathcal{L}-\mathcal{M}[f(t)] = \int_0^\infty t^{u-1}\ e^{-st}\ f(t)\ dt$	Mellin-Laplace Transform
(vi)	$\Im_{s,1}^{1,v}[f(t)]$	$\mathcal{L}-\mathcal{S}[f(t)] = \int_0^\infty \frac{e^{-st}}{(t+v)}\ f(t)\ dt$	Laplace-Stieltjes Transform
(vii)	$\Im_{0,1}^{0,v}[f(t)]$	$\mathcal{M}-\mathcal{S}[f(t)] = \int_0^\infty \frac{t^{u-1}}{(t+v)}\ f(t)\ dt$	Mellin-Stieltjes Transform

2.3 Properties of Ultra Gamma Transform

Some standard properties of ultra gamma transform, which all the classical transform possess, are enumerated here as a part of analysis:

2.3.1 Shifting/Translation Properties

The ultra gamma transform possesses three important shifting properties with respect to exponential function, power function and the binomial function respectively. The properties are as follows:

1. For $a > 0$, the first shifting property with respect to exponential function resulting in to shift in one of the parameters s is given by

$$\Im_{s,\lambda}^{u,v}[e^{at} f(t)] = \Im_{s-a,\lambda}^{u,v}[f(t)]; \tag{14}$$

2. For $b > 0$ and t^b being principal, the second shifting property with respect to power function resulting in to shift in another parameters u is given by

$$\Im_{s,\lambda}^{u,v}[t^b f(t)] = \Im_{s,\lambda}^{u+b,v}[f(t)]; \tag{15}$$

3. For $t \neq -d, m \in N$, the third shifting property with respect to binomial function resulting in to shift in u is expressed in terms of a finite series of shifted parameter ultra gamma transforms.

$$\Im_{s,\lambda}^{u,v}[(t+d)^m f(t)] = \sum_{n=0}^{m} C(m,n)\ d^{m-n}\ \Im_{s,\lambda}^{u+n,v}[f(t)]; \tag{16}$$

where $C(m,n)$ stands for usual binomial coefficients.

4. The property 1 and property 2 given above, when combined, give rise to a joint shifting property defined by

$$\Im_{s,\lambda}^{u,v}[e^{at}\ t^{b} f(t)] = \Im_{s-a,\lambda}^{u+b,v}[f(t)]. \tag{17}$$

This property may also be named as 2-D translation property with the fact that it translates one of the parameters in positive and the other in negative direction.

The proofs of all the four properties enumerated above are simple hence omitted.

2.3.2 Linearity Properties

If $f(t)$ and $g(t)$ be two locally integrable functions and a and b are two arbitrary numbers real or complex, then the ultra gamma transform possesses usual linearity property

$$\Im_{s,\lambda}^{u,v}[a\ f(t) \pm b\ g(t)] = a\ \Im_{s,\lambda}^{u,v}[f(t)] \pm b\ \Im_{s,\lambda}^{u,v}[g(t)], \tag{18}$$

which, for a finite sequence of functions $\langle f_n(t)\rangle$ and the finite sequence of real or complex numbers $\langle c_n\rangle$, can be further represented in more general form

$$\Im_{s,\lambda}^{u,v}[c_1 f_1(t) \pm \cdots \pm c_n f_n(t)] = c_1\ \Im_{s,\lambda}^{u,v}[f_1(t)] \pm \cdots \pm c_n \Im_{s,\lambda}^{u,v}[f_n(t)]. \tag{19}$$

2.3.3 Dual Scaling Property

Let $c > 1$ be a real number and $f(ct)$ sustains the integrability over the entire positive real line in Riemann sense. Then the property down-scales the parameter s and up-scales the parameter v and for $0 < c < 1$, the scaling is reversed between these parameters. Thus the name dual scaling property is justified. The property is read as follows:

$$\Im_{s,\lambda}^{u,v}[f(ct)] = \frac{1}{c^{u-\lambda}} \Im_{\frac{s}{c},\lambda}^{u,cv}[f(t)]. \tag{20}$$

This special dual scaling property of integral transform so defined makes its name *"Ultra Gamma Transform"* justified.

3. Relation with Fractional Integral Operator

Fractional calculus is the generalization of ordinary n-times iterated integrals and n^{th} derivatives of continuous functions to that of any arbitrary order real or complex. The most commonly used definition of fractional integral operators of order α is due to Riemann-Liouville. A detailed account of fractional calculus is given in Samko et al. [21] and the applications of it are elaborated in Podlubney [19]

and Hilfer [9]. Vyas [26], Vyas and Banerji ([27], [28]) and Vyas, Banerji and Saigo [29] have contributed to the application of fractional calculus in the evaluation of Dirichlet averages. Vyas [25] interpreted the angle of collision occurring in the study of transport properties of Noble gases at low density configuration using fractional integral operators of order $\frac{1}{2}$, i.e., semi-integrals. A detailed account of applications of semi-derivatives and semi-integrals to the problems of Electrical Engineering is given in McBride and Roach [15].
The operators of fractional integration of order α, $\Re(\alpha) > 0$ due to Riemann-Liouville and that of Weyl are respectively expressed as

$$ {}_0D_x^{-\alpha}[f(t)] = R_{0,x}^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\ dt, \tag{21} $$

$$ W_{x,\infty}^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\ dt, \tag{22} $$

and have tremendous applications in the solutions of fractional differential equations of Physics and Engineering.

Theorem 3: Let the UGT be defined by equation (12) under specified convergence conditions and $W_{x,\infty}^{\alpha}[f(t)],\ \Re(\alpha) > 0$ denotes the Weyl fractional integral operator of order α. Then the UGT can be represented in the following form

$$ \Im_{s,\lambda}^{u,v}[f(t)] = e^{sv}\ \Gamma(u)\ W_{v,\infty}^{u}\left[e^{-s\omega}\omega^{-\lambda}\ f(\omega - v)\right], \tag{23} $$

where $\Gamma(u)$ is the classical gamma function.
Proof: Making the linear substitution $t + v = \omega$ so that the limits of integration change from $(0, \infty)$ to (v, ∞), we write

$$ \Im_{s,\lambda}^{u,v}[f(t)] = \int_v^{\infty} \frac{(\omega - v)^{u-1} e^{-s(\omega - v)}}{\omega^{\lambda}}\ f(\omega - v)\ d\omega. $$

Rearranging the terms within the integral and invoking the definition of Weyl's fractional operator (22) we arrive at the desired result.

Corollary 1: Fractional integral representations of UGT and its particular cases suggest that the UGF and its particular cases can also be represented in terms of Weyl fractional integral. The UGF possesses following fractional integral representation. If we consider the function $f(t)$ to be unity, then the transform yields the UGF. Owing to the contents of theorem 3, the fractional integral representation of UGF is obtained in the following fashion

$$ \Im_{s,\lambda}^{u,v}[1] = \Gamma_{\lambda}(u, v, s) = e^{sv}\ \Gamma(u)\ W_{v,\infty}^{u}\left[e^{-s\omega}\omega^{-\lambda}\right], \tag{24} $$

which, for $s = 1$ & $\lambda = m$, further gives rise to fractional integral representation of Kobayashi's gamma function as a special case.

$$\Gamma_m(u, v, 1) = \Gamma_m(u, v) = e^v \ \Gamma(u) \ W^u_{v,\infty} \left[e^{-\omega} \omega^{-m} \right]. \tag{25}$$

Corollary 2: Using the fractional integral form of Kobayashi's generalized gamma function (25) for $u = n + \frac{1}{2}$, $v = -2i(\kappa + \lambda)$ & $m = 1$, the solution of integral equation appearing in Wiener-Hopf method for finite differaction may also be represented as

$$\Psi_+^{1,2}(\lambda) = \mp \sum_{n=0}^{N} \left(\frac{i}{2}\right)^n \frac{d^n \Psi_+^{1,2}(\kappa)}{n! \ d\lambda^n} \ \frac{\sqrt{2}}{\pi}(\kappa + \lambda) e^{-i(2\lambda - \pi/4)} \times$$

$$\Gamma\left(n + \frac{1}{2}\right) W^{n+\frac{1}{2}}_{-2i(\kappa+\lambda),\infty} \left[e^{-\omega} \omega^{-1} \right] + S^{1,2}(\lambda), \tag{26}$$

which indicates that the Weyl fractional integral finds an important role in describing the process of solution of such integral equation over a complex domain. A detailed account of fractional calculus on complex domain including equivalences is available in research monographs of Nishimoto [17].

4. Table of Ultra Gamma Transforms $\Im^{u,v}_{s,\lambda}[f(t)]$

A typical property of UGT is that any input function maps to ultra gamma function $\Gamma_\lambda(u, v, s)$ whose characteristic and asymptotic properties are addressed in Banerji and Sinha [5]. Also, the two parameter gamma function due to Kobayashi [13] follows as a particular case of three parameter gamma function, it is quite legitimate to discover the results involving integral transforms of functions in terms of Kobayashi's function $\Gamma_\lambda(u, v)$.

Present section deals with the ultra gamma transform of certain elementary functions in terms of ultra gamma function defined in the preceding sections. Following table lists the UGT of elementary functions t^n, e^{at} by direct evaluations and that of trigonometric functions, hyperbolic functions and the combinations thereof by using the properties of UGT in terms of UGF:

Table 2: Ultra Gamma Transforms in terms of Ultra Gamma Function

$\mathbf{f(t)}$	$\Im^{\mathbf{u,v}}_{\mathbf{s},\lambda}[\mathbf{f(t)}]$
1	$\Gamma_\lambda(u,v,s)$
t	$\Gamma_\lambda(u+1,v,s) = \Gamma'_\lambda(u,v,s)$
t^n	$\Gamma_\lambda(u+n,v,s) = \Gamma^{(n)}_\lambda(u,v,s)$
e^{at}	$\Gamma_\lambda(u,v,s-a)$
$t^n e^{at+b}$	$e^b\ \Gamma_\lambda(u+n,v,s-a) = e^b\Gamma^{(n)}_\lambda(u,v,s-a)$
$\sin at$	$\frac{1}{2i}\left[\Gamma_\lambda(u,v,s+ia) - \Gamma_\lambda(u,v,s-ia)\right]$
$\cos at$	$\frac{1}{2i}\left[\Gamma_\lambda(u,v,s+ia) + \Gamma_\lambda(u,v,s-ia)\right]$
$\sinh at$	$\frac{1}{2}\left[\Gamma_\lambda(u,v,s+a) - \Gamma_\lambda(u,v,s-a)\right]$
$\cosh at$	$\frac{1}{2}\left[\Gamma_\lambda(u,v,s+a) + \Gamma_\lambda(u,v,s-a)\right]$
$e^{at}\sin bt$	$\frac{1}{2i}\left[\Gamma_\lambda(u,v,s-a+ib) - \Gamma_\lambda(u,v,s-a-ib)\right]$
$e^{at}\cos bt$	$\frac{1}{2i}\left[\Gamma_\lambda(u,v,s-a+ib) + \Gamma_\lambda(u,v,s-a-ib)\right]$
$(t+p)^r$	$\sum_{n=0}^{r} C(r,n)\ p^{r-n}\ \Gamma_\lambda(u+n,v,s) = \sum_{n=0}^{r} C(r,n)\ p^{r-n}\ \Gamma^{(n)}_\lambda(u,v,s)$

4.1 Table of Generalized Gamma Transforms $\mathcal{D}^{u,v}_m[f(t)]$

The generalized gamma transform, defined in (13), of certain elementary functions, trigonometric and hyperbolic functions are provided in the following table:

Table 3: Generalized Gamma Transforms in terms of Kobayashi's function & UGF

$\mathbf{f(t)}$	$\mathcal{D}^{\mathbf{u,v}}_{\mathbf{m}}[\mathbf{f(t)}]$
1	$\Gamma_m(u,v)$
t	$\Gamma_m(u+1,v)$
t^n	$\Gamma_m(u+n,v)$
e^{at}	$\Gamma_m(u,v,1-a)$
$t^n e^{at+b}$	$e^b\ \Gamma_m(u+n,v,1-a)$
$\sin at$	$\frac{1}{2i}\left[\Gamma_m(u,v,1+ia) - \Gamma_m(u,v,1-ia)\right]$
$\cos at$	$\frac{1}{2i}\left[\Gamma_m(u,v,1+ia) + \Gamma_m(u,v,1-ia)\right]$
$\sinh at$	$\frac{1}{2}\left[\Gamma_m(u,v,1+a) - \Gamma_m(u,v,1-a)\right]$
$\cosh at$	$\frac{1}{2}\left[\Gamma_m(u,v,1-a) + \Gamma_m(u,v,1-a)\right]$
$e^{at}\sin bt$	$\frac{1}{2i}\left[\Gamma_m(u,v,1-a+ib) - \Gamma_m(u,v,1-a-ib)\right]$
$e^{at}\cos bt$	$\frac{1}{2i}\left[\Gamma_m(u,v,1-a+ib) + \Gamma_m(u,v,1-a-ib)\right]$
$(t+p)^r$	$\sum_{n=0}^{r} C(r,n)\ p^{r-n}\ \Gamma_m(u+n,v)$

The representation of $\mathcal{D}^{u,v}_m[f(t)]$ of elementary and other functions listed above in terms of UGF is the strong justification for defining the UGT. The generalized gamma transform of functions containing exponential and transcendental functions enjoys the definition of UGF defined in (1).

5. Probability Density Associated with UGF

In a systematic study of generalized probability density functions associated with generalized gamma-type functions as well as hypergeometric functions and their statistical properties, special functions find significant role ([3], [4], [10], [11], [16]). It is obvious that the gamma function (1) was studied by Banerji and Sinha [5] and may be derived from the generalized gamma-type functions defined by other authors. It is legitimate to present the following statistical affiliations of this function and to evaluate the bonafied properties in statistical analysis of pdf.
The probability density function (pdf) of a random variable X associated with ultra gamma function(1) is defined by,

$$f(x) = \frac{x^{u-1}e^{-sx}(x+v)^{-\lambda}}{\Gamma_\lambda(u,v,s)}, \quad 0 < x < \infty \tag{27}$$

It is very akin to observe that $\int_0^\infty f(x)\ dx = 1$, i.e., the total probability with respect to this pdf is unity. However, it is important to note that the nature of the above pdf at $x = 0$ can only be explained by considering the values of u, i.e.,

$$f(0) = \begin{cases} 0, & u > 1 \\ \Gamma_\lambda^{-1}(1,v,s), & u = 1 \end{cases} \tag{28}$$

5.1 Special Cases of $f(x)$

1. If we set $s = 1$ and $\lambda = m$, the density function becomes

$$f(x) = \frac{x^{u-1}e^{-x}(x+v)^{-m}}{\Gamma_m(u,v)}, \quad 0 < x < \infty, \tag{29}$$

   where $\Gamma_m(u,v)$ stands for Kobayashi's generalized gamma function defined in equation (9).

2. Taking $s = 1$ and $\lambda = 0$, the density function reduces to classical gamma distribution given by

$$f(x) = \frac{1}{\Gamma(u)}x^{u-1}e^{-x}, \quad 0 < x < \infty, \tag{30}$$

3. If we replace x by $\left(\frac{x}{\theta}\right)^{\beta}$, v by k and $s=1$, the pdf defined in (27) reshapes in the form of generalized gamma-type distribution recently considered by El-Fateh et al. [6] where they provide the application of the distribution in the mathematical modelling of inventory control problem. The distribution is expressed as

$$f(x;\alpha,k,\theta,\lambda,\beta)=\frac{\beta}{\theta\Gamma_{\lambda}(\alpha,k)}\left(\frac{x}{\theta}\right)^{\alpha\beta-1}e^{-\left(\frac{x}{\theta}\right)^{\beta}}\left[\left(\frac{x}{\theta}\right)^{\beta}+k\right]^{-m},\quad 0<x<\infty, \tag{31}$$

with $x>0$ and all the parameters are strictly positive except λ, which may be zero also. Also, $\Gamma_{\lambda}(\alpha,k)$ is already defined in equation (9).

5.2 Bonafied Statistical Properties

The aim of this section is to elaborate the distribution over a wide range of statistical properties that are useful in the analysis of a distribution in totality. These properties include moments, cumulative distribution function (cdf), moment generating function, etc.

5.2.1 The k-th Moment of $f(t)$

The k-th moment about origin of the scale of random variable X with respect to the pdf given by (27), is defined by

$$E[X^{k}]=\int_{0}^{\infty}t^{k}\ f(t)\ dt. \tag{32}$$

Inserting the value of the function in (32), we obtain

$$E[X^{k}]=\frac{\int_{0}^{\infty}t^{u+k-1}e^{-st}\ (t+v)^{\lambda}\ dt}{\Gamma_{\lambda}(u,v,s)}=\frac{\Gamma_{\lambda}(u+k,v,s)}{\Gamma_{\lambda}(u,v,s)}=\frac{\Gamma_{\lambda}^{(k)}(u,v,s)}{\Gamma_{\lambda}(u,v,s)}. \tag{33}$$

Now, one of the special cases of $E[X^{k}]$, for $k=1$, represents the mean of the random variable X, i.e., the mean is the first moment

$$E[X]=\int_{0}^{\infty}t\ f(t)\ dt=\frac{\Gamma_{\lambda}(u+1,v,s)}{\Gamma_{\lambda}(u,v,s)}=\frac{\Gamma_{\lambda}^{'}(u,v,s)}{\Gamma_{\lambda}(u,v,s)}. \tag{34}$$

Similarly, we can evaluate the variance of the random variable X, denoted by σ_{X}^{2}, using (32) with $k=2$ and the formula

$$\sigma_{X}^{2}=E[X^{2}]-(E[X])^{2}=\frac{\Gamma_{\lambda}^{'}(u,v,s)\Gamma_{\lambda}^{''}(u,v,s)}{\Gamma_{\lambda}^{2}(u,v,s)}\left[\frac{\Gamma_{\lambda}(u,v,s)}{\Gamma_{\lambda}^{'}(u,v,s)}-\frac{\Gamma_{\lambda}^{'}(u,v,s)}{\Gamma_{\lambda}^{''}(u,v,s)}\right]. \tag{35}$$

In equations (34) and (35), the prime and double prime over Γ denote the first and second partial derivatives of the UGF with respect to s respectively.
Note: The n^{th} derivative of generalized gamma function (UGF), with respect to s, proposed by Banerji and Sinha [5] at equation (3.11) seems to be incorrect, i.e., the formula given by them

$$\frac{\partial^n \Gamma_\lambda(u,v,s)}{\partial s^n} = (-)^n s^n \int_0^\infty \frac{t^{u-1}e^{-st}}{(t+v)^\lambda} dt, \tag{36}$$

needs to be corrected as

$$\frac{\partial^n \Gamma_\lambda(u,v,s)}{\partial s^n} = \Gamma_\lambda^{(n)}(u,v,s) = (-1)^n \int_0^\infty \frac{t^{u+n-1}e^{-st}}{(t+v)^\lambda} dt = (-1)^n \Gamma_\lambda(u+n,v,s). \tag{37}$$

5.2.2 The Moment Generating Function

The moment generating function of the random variable X is defined by

$$\mathcal{M}_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}\ f(x)\ dx. \tag{38}$$

Inserting the value of the function from (9) and combining the exponential terms in the integrands, we obtain the moment generating function given by

$$\mathcal{M}_X(t) = E[e^{tX}] = \frac{\Gamma_\lambda(u,v,s-t)}{\Gamma_\lambda(u,v,s)} = \Gamma_\lambda^{-1}(u,v,s)\Im_{s,\lambda}^{u,v}[e^{tx}], \tag{39}$$

where $\Im_{s,\lambda}^{u,v}[\cdots]$ is the UGT defined in equation (12) above. This generating function, upon using the Taylor series expansion preserving the convergence conditions, can also be written as

$$\mathcal{M}_X(t) = E[e^{tX}] = \Gamma_\lambda(u,v,s-t) = \sum_{k=0}^\infty \frac{t^k}{k!} E[X^k] = \sum_{k=0}^\infty \frac{\Gamma_\lambda^{(k)}(u,v,s)}{\Gamma_\lambda(u,v,s)} \frac{t^k}{k!}, \tag{40}$$

where $\Gamma_\lambda^{(0)}(u,v,s) = \Gamma_\lambda(u,v,s)$.

5.2.3 The Probability Distribution $\mathcal{F}(x)$

The cumulative density function $F(x)$ of a random variable associated with UGF is given by,

$$\mathcal{F}(x) = P(X \le x) = \int_0^x f(t)\ dt = \Gamma_\lambda^{-1}(u,v,s) \int_0^x \frac{t^{u-1}e^{-st}}{(t+v)^\lambda} dt = \frac{\Gamma_\lambda(u,v,s;x)}{\Gamma_\lambda(u,v,s)}, \tag{41}$$

where $\Gamma_\lambda(u,v,s;x)$ denotes the incomplete UGF studied by Banerji and Sinha [5]. Taking the cognizance of (41), the survivor function $\mathcal{S}(x)$ can be expressed as

$$\mathcal{S}(x) = P(X \geq x) = 1 - \mathcal{F}(x) = \int_x^\infty f(t)\ dt = \frac{\gamma_\lambda(u,v,s;x)}{\Gamma_\lambda(u,v,s)}. \tag{42}$$

Here $\gamma_\lambda(u,v,s;x)$ denotes another incomplete gamma function, of which the special cases / generalizations have been defined and studied by many mathematicians in applicable analysis.

5.2.4 The Hazard Rate Function $h(x)$

The hazard rate function for a probability density function $f(x)$ is defined by

$$h(x) = \frac{f(x)}{\mathcal{S}(x)} \tag{43}$$

Using (27) and (42), it follows that

$$h(x) = \frac{x^{u-1}e^{-sx}(x+v)^{-\lambda}}{\gamma_\lambda(u,v,s;x)}. \tag{44}$$

The hazard function for probability density consisting of Kobayashi's generalized gamma function and that for classical gamma distribution can be obtained by specializing the appropriate parameter(s) involved in (41).

5.2.5 The Mean Residue Life Function $\mathcal{K}(x)$

For a random variable X, the mean residue life function is defined by

$$\mathcal{K}(x) = E[X - x/X \geq x] = \frac{\int_x^\infty (t-x)\ f(t)\ dt}{\mathcal{S}(x)} = \frac{\int_x^\infty t\ f(t)\ dt}{\mathcal{S}(x)} - x. \tag{45}$$

Now, since $\int_x^\infty f(t)\ dt = \gamma_\lambda(u,v,s;x)/\Gamma_\lambda(u,v,s)$, it follows

$$\mathcal{K}(x) = \frac{\gamma_\lambda(u+1,v,s;x)}{\Gamma_\lambda(u,v,s)} - x. \tag{46}$$

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Appendix

H-function:

The Fox's H-function is defined by following contour integral representation

$$H_{p,q}^{m,n}\left[\begin{array}{c}(a_p, A_p)\\(b_q, B_q)\end{array}\middle| z\right] = H_{p,q}^{m,n}\left[\begin{array}{c}(a_1, A_1), \cdots, (a_p, A_p)\\(b_1, B_1), \cdots, (b_q, B_q)\end{array}\middle| z\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} z^s \chi(s)\, ds, \quad (47)$$

where $i = (-1)^{1/2}, z \neq 0$ and $z^s = \exp[s \log|z| + i\arg z]$, in which $\log|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value. An empty product is interpreted as unity. Here $\chi(s)$ stands for

$$\chi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)}, \quad (48)$$

where m, n, p and q are nonnegative integers such that $0 \leq n \leq p$, $1 \leq m \leq q$; $A_j (j = 1, \cdots, p)$, $B_j (j = 1, \cdots, q)$ are positive numbers; $a_j (j = 1, \cdots, p)$, $b_j (j = 1, \cdots, q)$ are complex numbers such that

$$A_j(b_h + \nu) \neq B_h(a_j - \lambda - 1),$$

for $\nu,\ \lambda = 0, 1, 2, \ldots;\ h = 1, \ldots, m;\ j = 1, \ldots, n.$
For description of the contour $\mathcal{L}$ and other conditions of existence of this function, one may refer Mathai and Saxena [14].
Following properties of H-function have been used in the findings of this paper:

Property 1: Argument reversion property of the H-function is

$$H_{p,q}^{m,n}\left[\begin{array}{c}(a_p, A_p)\\(b_q, B_q)\end{array}\middle| x\right] = H_{p,q}^{m,n}\left[\begin{array}{c}(1 - b_q, B_q)\\(1 - a_p, A_p)\end{array}\middle| \frac{1}{x}\right]. \quad (49)$$

This is the most important property which enables to transform the H-function from its argument x to the argument $1/x$.

Property 2: Multiplication of H-function with power function yields another H-function given by

$$x^{\sigma} H_{p,q}^{m,n}\left[\begin{array}{c}(a_p, A_p)\\(b_q, B_q)\end{array}\middle| x\right] = H_{p,q}^{m,n}\left[\begin{array}{c}(a_p + \sigma A_p, A_p)\\(b_q + \sigma B_q, B_q)\end{array}\middle| x\right]; \quad (50)$$

Property 3: Following result pertaining to a special case of H-function as binomial function has also been used in the sequel of this paper

$$H_{1,1}^{1,1}\left[\begin{array}{c}(1-\nu,1)\\(0,1)\end{array}\middle| x\right]=\Gamma(\nu)(1+x)^{-\nu}. \tag{51}$$

Wiener-Hopf technique:

The integral equation appearing in Wiener-Hopf technique related to mathematical theory of diffraction by a finite strip is

$$\frac{\Psi_+^{1,2}(\lambda)}{\sqrt{\kappa+\lambda}}\pm\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic}\frac{\Psi_+^{1,2}(\lambda)}{\sqrt{\alpha-\kappa}}\frac{e^{2i\alpha}}{\alpha+\lambda}\,d\alpha \;=\; F_a(\lambda)+G_a(\lambda)\pm F_b(\lambda)\pm G_b(\lambda), \tag{52}$$

where

$$F_{a,b}(\lambda)=\frac{qe^{2ip}}{\lambda\pm p}\left(\frac{1}{\sqrt{\kappa+\lambda}}-\frac{1}{\sqrt{\kappa\mp\lambda}}\right),\quad G_{a,b}(\lambda)=-\frac{qe^{2ip}}{\lambda\pm p}\left(W_0(\lambda)-W_0(\mp p)\right)$$

and

$$-\kappa_2\cos\theta_i<-c<\mathrm{Im}(\lambda)<c<\kappa_2\cos\theta_i,\quad W_0(\lambda)=\sqrt{2}e^{-i(2\lambda+\pi/4)}\frac{F(2\sqrt{(\kappa+\lambda)/\pi})}{\sqrt{\kappa+\lambda}}$$

and the Fresnel integral is defined by

$$F(z)=\int_z^{\infty}e^{\frac{i\pi}{2}t^2}dt. \tag{53}$$