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REDUCTION FORMULA ASSOCIATED WITH WHITTAKER FUNCTION

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Dedicated to Prof. M.A. Pathan on his 75^{th} ***birth anniversary***

Abstract: Significant result is obtained in the present study in terms of reduction formulas of Srivastava's function $F^{(3)}$ into a combination of Exton's double hypergeometric function $X^{A:B;D}_{E:G;H}$. We then make use of our main result to derive a number of known and new transformation and reduction formulas for some Srivastava's triple hypergeometric series, Exton double hypergeometric function, Appell function etcetera.

Keywords: Whittaker function, Preece result, Appell's, Exton and Srivastava's triple hypergeometric series and Pochhammer symbol.

2010 Mathematics Subject Classification: 33C20, 33C65, 33C70, 33C80, 33C90.

1. Introduction

Saran [16], Sivastava [11], Exton [9], Srivastava and Karlsson [14] have discussed many transformations and interesting instances of the reducibility of triple hypergeometric functions. These results are obtained mainly by manipulations of the series. The study of transformation and reduction formulae have occupied the attention of many authors. The searching technique of the manipulations of the series has classically found wide application in this field. It is now employed together with Preece result in the present paper to obtain the main reduction formula of Srivastava's function $F^{(3)}$ into a combination of Exton's double hypergeometric function $X^{1:2;1}_{0:3;1}$. Some deduction from this formula lead us to a number of known and new transformation and reduction formulas for some Exton's double hypergeometric function and Appell function F_2.

In the literature of special functions [11, 5, 15, 14, 13], the Kummer's formula related with single Gaussian hypergeometric function play an important role in the study of transformations and reduction formula of multiple Gaussian hypergeometric functions and some integrals are include in section (2.2).

A natural generalization of the hypergeometric function ${}_2F_1$ is the generalized hypergeometric function, so called ${}_pF_q$ which is defined as

$$ {}_pF_q\left[\begin{array}{r} a_1,\ldots\ldots,a_p; \\ z \\ b_1,\cdots\cdots,b_q; \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} \tag{1.1} $$

where, as usual

$$ (a_i)_n = \frac{\Gamma(a_i+n)}{\Gamma(a_i)} \quad \text{and} \quad [(a)]_n = \prod_{i=1}^{p}(a_i)_n. $$

Here p and q are positive integers or zero, the numerator parameters $a_1,\cdots,a_p$ and the denominator parameters $b_1,\ldots,b_q$ take on complex values, provided that $b_j \neq 0,-1,-2,\ldots;\ j=1,2,\ldots q$.

Thus, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates. Suppose that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), then the ${}_pF_q$ series in (1.1)

(a) converges for $\mid z \mid < \infty$ if $p \leq q$,

(b) converges for $\mid z \mid < 1$ if $p \leq q$, and

(c) diverges for all z, $z \neq 0$, if $p > q+1$.

(d) if $p = q+1$, the series in (1.1) is absolutely convergent on the circle $\mid z \mid = 1$, if

$$ \mathrm{Re}\left(\sum_{\mathrm{j}=1}^{\mathrm{q}} \mathrm{b_j} - \sum_{\mathrm{i}=1}^{\mathrm{p}} \mathrm{a_i}\right) > 0. $$

The general triple hypergeometric function $F^{(3)}$ of Srivastava [12; p. 428] is the unification and generalization of Lauricella's fourteen hypergeometric functions of three variables and the additional functions H_A, H_B, H_C was introduced by Srivastava in the form of a general triple hypergeometric series $F^{(3)}[x,y,z]$ defined as:

$$ F^{(3)}\left[\begin{array}{lc} (a_A) :: (b_B); (d_D); (e_E) : (g_G); (h_H); (l_L); & \\ & x,y,z \\ (m_M) :: (n_N); (p_P); (q_Q) : (r_R); (s_S); (t_T); & \end{array}\right] $$

$$= \sum_{i,j,k=0}^{\infty} \frac{[(a_A)]_{i+j+k}[(b_B)]_{i+j}[(d_D)]_{j+k}[(e_E)]_{k+i}[(g_G)]_i[(h_H)]_j[(l_L)]_k}{[(m_M)]_{i+j+k}[(n_N)]_{i+j}[(p_P)]_{j+k}[(q_Q)]_{k+i}[(r_R)]_i[(s_S)]_j[(t_T)]_k} \frac{x^i y^j z^k}{i!j!k!}. \quad (1.2)$$

In 1982, Exton [8; p. 137 (1.2)] defined the following double hypergeometric function

$$X^{A:B:D}_{E:G:H}\left[\begin{array}{l} (a_A);(b_B);(d_D); \\ \\ (e_E);(g_G);(h_H); \end{array} x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[(a_A)]_{2m+n}[(b_B)]_m[(d_D)]_n}{[(e_E)]_{2m+n}[(g_G)]_m[(h_H)]_n} \frac{x^m}{m!}\frac{y^n}{n!}, \quad (1.3)$$

which is the generalization and unification of Horn's non confluent double hypergeometric function H_4 [2; p. 225 (16)] and Horn's confluent double hypergeometric function H_7 [2; p. 226 (35)].

2. Some Useful Standard Results

Here we mention the following standard results, which are used to obtained our main result.

(i) The integrals representations with Whittaker's function [1; p. 215(11)], [3; p. 255(1)]

$$\int_0^\infty e^{-pt} t^{A-1} M_{k,\mu}(\alpha t)\, dt$$

$$= \frac{\Gamma(A+\mu+\frac{1}{2})\alpha^{\mu+\frac{1}{2}}}{(p+\frac{\alpha}{2})^{A+\mu+\frac{1}{2}}} \, {}_2F_1\left[\begin{array}{ll} A+\mu+\frac{1}{2}, \mu-k+\frac{1}{2} & ; \\ & \\ 2\mu+1 & ; \end{array} \frac{\alpha}{(p+\frac{\alpha}{2})}\right] \quad (2.1)$$

$$= \frac{\Gamma(A+\mu+\frac{1}{2})2^{A+\mu+\frac{1}{2}}\alpha^{\mu+\frac{1}{2}}}{(2p-\alpha)^{A+\mu+\frac{1}{2}}} \, {}_2F_1\left[\begin{array}{ll} A+\mu+\frac{1}{2}, \mu+k+\frac{1}{2} & ; \\ & \\ 2\mu+1 & ; \end{array} \frac{2\alpha}{(\alpha-2p)}\right], \quad (2.2)$$

where $\text{Re}(\text{A}+\mu) > -\frac{1}{2}$; $2\text{Re}(\text{p}) >| \alpha |$ or $\text{Re}(\rho - \text{A})$, $2\text{Re}(\text{p}) = \text{Re}(\alpha) > 0$, and $M_{k,\mu}(x)$ is a whittaker function [5] defined as

$$M_{k,\mu}(x) = x^{\mu+1/2} e^{(-1/2)x} \, {}_1F_1\left[\begin{array}{l} \frac{1}{2}+\mu-k; \\ \\ 2\mu+1; \end{array} x\right]. \quad (2.3)$$

(ii) Preece result [4; p. 378(11)]

$$ {}_1F_1(\alpha;\rho;x)\ {}_1F_1(\alpha-\rho+1;2-\rho;-x) = {}_2F_3\left(\begin{array}{ll} \frac{1}{2}-\frac{\rho}{2}+\alpha, \frac{1}{2}+\frac{\rho}{2}-\alpha \ ; & \\ & \frac{x^2}{4} \\ \frac{1}{2}+\frac{\rho}{2},\ \frac{1}{2},\ \frac{3}{2}-\frac{\rho}{2} \ ; & \end{array}\right) $$

$$ + \frac{(2\alpha-\rho)(1-\rho)x}{(2-\rho)\rho}\ {}_2F_3\left(\begin{array}{ll} 1-\frac{\rho}{2}+\alpha, 1+\frac{\rho}{2}-\alpha \ ; & \\ & \frac{x^2}{4} \\ 1+\frac{\rho}{2},\ 2-\frac{\rho}{2},\ \frac{3}{2} \ ; & \end{array}\right). \tag{2.4} $$

(iii) Kummer's first formula [12; p. 37(7)]

$$ {}_1F_1\left[\begin{array}{ll} a \ ; & \\ & x \\ c \ ; & \end{array}\right] = e^x\ {}_1F_1\left[\begin{array}{ll} c-a \ ; & \\ & -x \\ c \ ; & \end{array}\right], \quad c \neq 0,-1,-2,.... \tag{2.5} $$

3. Main reduction formula and its proof.

This section deals with main reduction formula of Srivastava's triple hypergeometric series $F^{(3)}$ into a combination of Exton's double hypergeometric function:

$$ F^{(3)}\left[\begin{array}{ll} a :: \_\_\_ ; \_\_\_ ; \_\_\_ ; \rho-\alpha \ ; 1-\alpha; c-b \ ; & \\ & \frac{2z}{1+y}, \frac{-2z}{1+y}, \frac{2y}{1+y} \\ \_\_\_ :: \_\_\_ ; \_\_\_ ; \_\_\_ ; \rho; 2-\rho; a-b+1 \ ; & \end{array}\right] $$

$$ = \left(\frac{1+y}{1-y}\right)^a \Bigg[\ X^{1:2:1}_{0:3:1}\left(\begin{array}{ll} a :: \frac{1+\rho-2\alpha}{2},\ \frac{1-\rho+2\alpha}{2},\ a-c+1 \ ; & \\ & \frac{z^2}{(1-y)^2}, \frac{2y}{y-1} \\ \_\_\_ :: \frac{1}{2},\ \frac{1+\rho}{2},\ \frac{3-\rho}{2};\ a-b+1 \ ; & \end{array}\right) $$

$$ + \frac{4ayz(\rho-1)(\rho-2\alpha)}{(1-y)\rho(2-\rho)} $$

$$ X^{1:2:1}_{0:3:1}\left(\begin{array}{ll} a+1 :: \frac{2-\rho+2\alpha}{2};\ \frac{2+\rho-2\alpha}{2};\ a-c+2 \ ; & \\ & \frac{z^2}{(1-y)^2}, \frac{2y}{y-1} \\ \_\_\_ :: \frac{3}{2},\ \frac{2+\rho}{2},\ \frac{4-\rho}{2};\ a-b+2 \ ; & \end{array}\right) \Bigg]. \tag{3.1} $$

Proof of (3.1).

In order to obtain the main reduction formula of this paper, we establish an integral in the following form:

$$\mathrm{I} = \int_0^\infty e^{-t} t^b M_{k,\mu}(pt) dt \ {}_1F_1 \left[\begin{matrix} \rho - \alpha; \\ \rho; \end{matrix} \ -xt \right] {}_1F_1 \left[\begin{matrix} 1-\alpha; \\ 2-\rho; \end{matrix} \ xt \right] dt. \tag{3.2}$$

In the above integral expanding both ${}_1F_1$ into power series and using the result (2.1), we get

$$\mathrm{I} = \frac{\Gamma(b+\mu+3/2) p^{\mu+1/2}}{(1+p/2)^{b+\mu+3/2}}$$

$$F^3 \left[\begin{matrix} b+\mu+3/2 :: -;-;-; \rho-\alpha; 1-\alpha; \mu-k+\frac{1}{2}; \\ - :: -;-;-; \quad \rho; 2-\rho; 2\mu+1; \end{matrix} \ \frac{-2x}{2+p}, \frac{2x}{2+p}, \frac{2p}{2+p} \right], \tag{3.3}$$

by the application of Kummer's first formula (2.4) and preece result (2.3), equation (3.2) reduces to

$$\mathrm{I} = \int_0^\infty e^{-t} t^b M_{k,\mu}(pt) \ {}_2F_3 \left[\begin{matrix} \frac{1+\rho-2\alpha}{2}, \frac{1-\rho+2\alpha}{2}; \\ \frac{1}{2}, \frac{1+\rho}{2}, \frac{3-\rho}{2}; \end{matrix} \ \frac{x^2t^2}{4} \right] dt$$

$$+ \frac{(\rho-1)(\rho-2\alpha)x}{\rho(2-\rho)} \int_0^\infty e^{-t} t^b M_{k,\mu}(pt) \ {}_2F_3 \left[\begin{matrix} \frac{2-\rho+2\alpha}{2}, \frac{2+\rho-2\alpha}{2}; \\ \frac{3}{2}, \frac{2+\rho}{2}, \frac{4-\rho}{2}; \end{matrix} \ \frac{x^2t^2}{4} \right] dt. \tag{3.4}$$

On expanding ${}_2F_3$ in a power series and then using the result (2.2), we get

$$\mathrm{I} = 2^{b+\mu+3/2} \frac{\Gamma(b+\mu+3/2) p^{\mu+\frac{1}{2}}}{(2-p)^{b+\mu+\frac{3}{2}}}$$

$$X_{0:3:1}^{1:2:1} \left[\begin{matrix} b+\mu+\frac{3}{2} : \frac{1+\rho-2\alpha}{2}, \frac{1-\rho+2\alpha}{2} \ ; \mu+k+\frac{1}{2}; \\ - :: \frac{1}{2}, \quad \frac{1+\rho}{2}, \quad \frac{3-\rho}{2}, \quad 2\mu+1; \end{matrix} \ \frac{x^2}{(2-p)^2}, \frac{2p}{p-2} \right]$$

$$+ \frac{x(\rho-1)(\rho-2\alpha)\Gamma(b+\mu+\frac{5}{2}) 2^{b+\mu+\frac{5}{2}} p^{\mu+\frac{3}{2}}}{\rho(2-\rho) \quad (2-p)^{b+\mu+\frac{5}{2}}}$$

$$X_{0:3:1}^{1:2:1}\left[\begin{array}{l} b+\mu+5/2 : \frac{2-\rho+2\alpha}{2}, \frac{2+\rho-2\alpha}{2}; \mu+k+3/2; \\ : - : \frac{3}{2}, \quad \frac{2+\rho}{2}, \quad \frac{4-\rho}{2};\ 2\mu+2; \end{array} \frac{x^2}{(2-p)^2}, \frac{2p}{p-2}\right]. \quad (3.5)$$

Equating (3.3) and (3.5), adjusting the parameters and replacing μ, b, k, x and p by $\frac{a-b}{2}$, $\frac{a+b-3}{2}$, $\frac{a+b-2c+1}{2}$, $2z$ and $2y$, respectively, we get the main reduction formula (3.1).

4. Special Cases.

In this section, we deduce some known and new reduction formulas for hypergeometric function, Appell function F_2 and Exton's double hypergeometric function.

(i) Taking $y = 0$ and z is replace by $\frac{z}{2}$ in (3.1), we get

$$F_2\left[a, \rho-\alpha, 1-\alpha; \rho, 2-\rho; z, -z\right] = {}_4F_3\left[\begin{array}{l} \frac{a}{2}, \frac{a+1}{2}, \frac{1+\rho-2\alpha}{2}, \frac{1-\rho+2\alpha}{2} \quad ; \\ \frac{1}{2}, \frac{1+\rho}{2}, \frac{3-\rho}{2} \quad ; \end{array} z^2\right]. \quad (4.1)$$

where F_2 is Appell's function of second kind [12].

(ii) Taking $\rho = 2\alpha$ in (3.1), we get

$$F^{(3)}\left[\begin{array}{l} a :: -; -; -; \alpha, 1-\alpha; c-b; \\ - :: -; -; -; 2\alpha, 2-2\alpha; a-b+1; \end{array} \frac{2z}{1+y}, \frac{-2z}{1+y}, \frac{2y}{1+y}\right]$$

$$= \left(\frac{1+y}{1-y}\right)^a X_{0:2:1}^{1:1:1}\left[\begin{array}{l} a, \frac{1}{2}, a-c+1; \\ - :: \frac{1+2\alpha}{2}, \frac{3-2\alpha}{2}; a-b+1; \end{array} \frac{z^2}{(1-y)^2}, \frac{2y}{y-1}\right]. \quad (4.2)$$

(iii) Taking $\rho = 2\alpha$ in (4.1), we get

$$F_2\left[a, \alpha, 1-\alpha; 2\alpha, 2-2\alpha; z, -z\right] = {}_3F_2\left[\begin{array}{l} \frac{a}{2}, \frac{a+1}{2}, \frac{1}{2} \quad ; \\ \frac{1+2\alpha}{2}, \frac{3-2\alpha}{2} \quad ; \end{array} z^2\right]. \quad (4.3)$$

(iv) Taking $z = 1$ in (4.3), we get

$$F_2\left[a, \alpha, 1-\alpha; 2\alpha, 2-2\alpha; 1, -1\right] = {}_3F_2\left[\begin{array}{l} \frac{a}{2}, \frac{a+1}{2}, \frac{1}{2} \quad ; \\ \frac{(1+2\alpha)}{2}, \frac{(3-2\alpha)}{2} \quad ; \end{array} 1\right]. \quad (4.4)$$

(v) Taking $z = 0$ in (4.2), we get a known Euler's transformation [13; p. 33(19)]

$$\left(\frac{1+y}{1-y}\right)^{-a} {}_2F_1\left(\begin{array}{lr} a, c-b & ; \\ & \frac{2y}{1+y} \\ a-b+1 & ; \end{array}\right) = {}_2F_1\left(\begin{array}{lr} a, a-c+1 & ; \\ & \frac{2y}{y-1} \\ a-b+1 & ; \end{array}\right). \tag{4.5}$$

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