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APPROXIMATE SOLUTIONS OF THE D-DIMENSIONAL SCHRODINGER EQUATION WITH MULTIPARAMETER-TYPE POTENTIAL USING NIKIFOROV-UVAROV METHOD

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Dedicated to Prof. M.A. Pathan on his 75^{th} birth anniversary

Abstract: The analytical solution to the Schrödinger equation in D-dimensions with multiparameter-type potential were obtained using Nikiforov-Uvarov method, and applying the pekeris approximation to the centrifugal term. For convenience,the equation are reduced to the hypergeometricform,where the energy eigen values and corresponding eigenfunction are obtained.The expectation values $\langle r^{-2}\rangle$, $\langle q+e^{2\alpha r}\rangle^{-1}$ and $\langle q+e^{2\alpha r}\rangle^{-2}$ are obtained in D-Dimension using Hellmann-Feynman Theorem.

Keywords: Schrodinger equation, multiparameter potential, Nikiforov-Uvarov method, Hellmann-Feynman Theorem.

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1. Introduction

The Schrodinger wave equation acts as the engine room of quantum mechanics. over the years and in recent times the exact solution of Schrödinger equation for some special physical potential has attracted so much interest. Some of these potential are the Hulthen potential, [1] the Rose- Morse potential, [2] the Morse potential, [3] the Eckart potential, [4] the five parameter exponential-type potential, [5] the Poschl-Teller potential, [6] Manning-Rosen potential [7] and others. The harmonic oscillator and Hydrogen atom problems are two exactly solvable potentials which have been investigated in N-dimensional quantum mechanics for any angular momentum . [8-10] These two problems are related and hence the resulting in second order differential equation has the normalized orthogonal polynomial function solution. [11] The analytical method have also been used to solve the wood-saxon and manning-rosen potential [12]. Different methods have been introduced in solving schrödinger equation for various potentials [13]. Among such methods include the super symmetric(SUSY)and shape-invariance method, [14] the variational, [15]the standard method, [16] path integral approach, [17] the asymptotic interaction method (AIM), [18] the Nikiforov-Uvarov method(NU) [19] and others. In this work we have tried to investigate the analytical solution of the schrodinger equation in D-dimension with Multiparameter-type potential [20-21].

$$V_{(r)} = A + \frac{B}{(q + e^{2\alpha r})} + \frac{C}{(q + e^{2\alpha r})^2} + \frac{Fbe^{2\alpha r}}{(q + e^{2\alpha r})} + \frac{Gbe^{2\alpha r}}{(q + e^{2\alpha r})^2} \tag{1}$$

where A, B, C, F, G are the potential parameters,q is the deformation parameter, $b = e^{2\alpha r_e}$, r_e is the distance form equilibrium position and α is the screening parameter, which have diverse application in physics, chemistry and applied mathematics.

2. Review of Nikiforov-Uvarov

The Nikiforov-Uvarov method [22] was introduced to solve second-order differential equation of the form

$$\varphi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\varphi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\varphi(s) = 0 \tag{2}$$

With appropriate co-ordinate transformation, $s = s(r)$, where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most a second order and $\tilde{\tau}(s)$ is a first degree polynomial. The parametric form of the schrödinger-like equation is stated as [23]

$$\frac{d^2\varphi}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{d\varphi}{ds} + \frac{1}{s^2(1-\alpha_3 s)^2}[-\xi_1 s^2 + \xi_2 s - \xi_3]\varphi(s) = 0 \tag{3}$$

The eigenfunction and the corresponding energy eigenvalues equation are obtained according to NU method [24]

$$\varphi(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} p_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{4}$$

$$(\alpha_2-\alpha_3)n + \alpha_3 n^2 - (2n+1)\alpha_5 + (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0 \tag{5}$$

$$\alpha_4 = \frac{1-\alpha_1}{2}, \quad \alpha_5 = \frac{(\alpha_2 - 2\alpha_3)}{2}, \quad \alpha_6 = \alpha_5^2 + \xi_1, \quad \alpha_7 = 2\alpha_4\alpha_5 - \xi_2,$$

$$\alpha_8 = \alpha_4^2 + \xi_3, \quad \alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6, \quad \alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8},$$

$$\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),\quad \alpha_{12}=\alpha_4+\sqrt{\alpha_8},\quad \alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}) \tag{6}$$

3 D-Dimensional Schrödinger Equation and Solutions

The Schrödinger equation for a spherically symmetric potential in D-dimensional [25] reads

$$\frac{-\hbar^2}{2\mu}[\nabla_D^2+V(r)]\psi_{nlm}(r,\Omega_m)=E_{nl}\psi_{nlm}(r,\Omega_m), \tag{7}$$

where the Laplacian operator is defined as

$$\nabla_D^2=\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left[r^{D-1}\frac{\partial}{\partial r}\right]-\frac{\Lambda_D^2(\Omega_D)}{r^2}, \tag{8}$$

where $V(r)$ is the potential, μ is the reduced mass, $\hbar$ is the reduced planck constant, E_{nl} is the energy spectrum and Ω_D represents the angular co-ordinate. The hyperspherical harmonic functions are the eigenfunction of the operator $\Lambda_D^2(\Omega_D)$. Thus, we write

$$\psi_{nlm}(r,\Omega_m)=R_{nl}(r)Y_l^m(\Omega_D) \tag{9}$$

Where $Y_l^m(\Omega_D)$ are hypespherical harmonic and $R_{nl}(r)$ is the hyper radial wave function. It is well known that $\frac{\Lambda_D^2(\Omega_D)}{r^2}$ is a generalization of the centrifugal barrier for the D-dimensional space and involve the angular co-ordinate (Ω_D) and the eigenvalue of the hyperspherical harmonic functions $\Lambda_D^2(\Omega_D)$ are given by

$$\Lambda_D^2(\Omega_D)Y_l^m(\Omega_D)=l(l+D-2)Y_l^m(\Omega_D) \tag{10}$$

where l is the arbitrary angular momentum quantum number. By choosing a common ansatz for the wave function in the form

$$R_{nl}(r)=r^{-(D-1)/2}U_{nl}(r) \tag{11}$$

Equation (7) reduces into the Schrodinger equation in D-Dimension as [25]

$$\frac{d^2U_{nl}(r)}{dr^2}+\frac{2\mu}{\hbar^2}[E-V(r)]U_{nl}(r)+\frac{1}{r^2}\left[\frac{(D-1)(D-3)}{4}+l(l+D-2)\right]U_{nl}(r)=0 \tag{12}$$

Substituting equation (1) into (12), we have

$$\frac{d^2U_{nl}(r)}{dr^2}+\left\{\frac{2\mu E}{\hbar^2}-\frac{2\mu}{\hbar^2}\left[A+\frac{B}{(q+e^{2\alpha r})}+\frac{C}{(q+e^{2\alpha r})}+\frac{Fbe^{2\alpha r}}{(q+e^{2\alpha r})}+\frac{Gbe^{2\alpha r}}{(q+e^{2\alpha r})^2}\right]\right.$$

$$+\frac{1}{r^2}\left(\frac{(D-1)(D-3)}{4}+l(l+D-2)\right)\right\} U_{nl}(r)=0 \tag{13}$$

To solve equation (13) for $l \neq 0$ we need to apply the Pekeris approximation to the centrifugal term given by [26]

$$\frac{1}{r^2} \approx \frac{1}{r_e^2}\left(d_0+d_1\frac{1}{(q+e^{2\alpha r})}+d_2\frac{1}{(q+e^{2\alpha r})^2}\right) \tag{14}$$

Introducing a change in variable through $s=-qe^{-2\alpha r}$, We obtain the following compact hyper geometric equation

$$\frac{d^2U_{nl}}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dU_{nl(s)}}{ds}+\frac{1}{s^2(1-s)^2}[-(\varepsilon^2+\gamma+\beta)s^2$$

$$+(2\varepsilon^2+\gamma+m)s-(\varepsilon^2+\varphi)]U_{nl(s)}=0 \tag{15}$$

where

$$-\varepsilon^2=\frac{2\mu E}{4\alpha^2\hbar^2}-\frac{2\mu A}{4\alpha^2\hbar^2}+d_0\frac{1}{4\alpha^2 r_e^2}\left[\frac{(D-1)(D-3)}{4}+l(l+D-2)\right] \tag{16}$$

$$\gamma=\frac{2\mu B}{4\alpha^2\hbar^2 q}-d_1\frac{1}{4\alpha^2 r_e^2 q}\left[\frac{(D-1)(D-3)}{4}+l(l+D-2)\right] \tag{17}$$

$$\beta=\frac{2\mu C}{4\alpha^2\hbar^2 q}-d_2\frac{1}{4\alpha^2 r_e^2 q^2}\left[\frac{(D-1)(D-3)}{4}+l(l+D-2)\right] \tag{18}$$

$$M=\frac{2\mu Fb}{4\alpha^2\hbar^2}+\frac{2\mu Gb}{4\alpha^2\hbar^2} \tag{19}$$

$$\varphi=\frac{2\mu Fb}{4\alpha^2\hbar^2} \tag{20}$$

Now comparing equation (15) with equation (3), we find the following parameter

$$\alpha_1=\alpha_2=\alpha_3=1,\quad \xi_1=\varepsilon^2+\gamma+\beta,\quad \xi_2=2\varepsilon^2+\gamma+M,\quad \xi_3=\varepsilon^2+\varphi \tag{21}$$

Using equation (6), We determine the remaining co-efficient as

$$\alpha_4=0,\quad \alpha_5=-\frac{1}{2},\quad \alpha_6=\varepsilon^2+\gamma+\beta+\frac{1}{4},$$

$$\alpha_7 = -2\varepsilon^2 - \gamma - M, \quad \alpha_8 = \varepsilon^2 + \varphi, \quad \alpha_9 = \varphi + \beta + \frac{1}{4} - M,$$

$$\alpha_{10} = 1 + 2\sqrt{\varepsilon^2 + \varphi}, \quad \alpha_{11} = 2 + 2\left(\sqrt{\varphi + \beta + \frac{1}{4} - m} + \sqrt{\varepsilon^2 + \varphi}\right)$$

$$\alpha_{12} = \sqrt{\varepsilon^2 + \varphi}, \quad \alpha_{13} = -\frac{1}{2} - \left(\sqrt{\varphi + \beta + \frac{1}{4} - M} + \sqrt{\varepsilon^2 + \varphi}\right) \tag{22}$$

Using equation (22) the explicit form of the energy eigenvalues is given as

$$E = -\frac{2\alpha^2\hbar^2}{\mu}\left\{\left[\frac{\varphi - \beta - \gamma}{2(n+\sigma)} + \frac{(n+\sigma)}{2}\right]^2 + \right.$$

$$\left. d_0\frac{1}{4\alpha^2 r_e^2}\left[\frac{(D-1)(D-3)}{4} + l(l+D-2)\right]\right\} + Fb + A \tag{23}$$

where,

$$\sigma = \frac{1}{2} + \sqrt{\varphi + \beta + \frac{1}{4} - M} \tag{24}$$

and the explicit wave function is given as

$$\varphi(r) = (-qe^{-2\alpha r})^{\sqrt{\varepsilon^2+\varphi}}(1+qe^{-2\alpha r})^{\frac{1}{2}+\sqrt{\varphi+\beta+\frac{1}{4}-M}}P_n^{(2\sqrt{\varepsilon^2+\varphi},2\sqrt{\varphi+\beta+\frac{1}{4}-M})}(1+2qe^{-2\alpha r}) \tag{25}$$

The expectation values $\langle r^{-2}\rangle$, $\langle q + e^{2\alpha r}\rangle^{-1}$ and $\langle q + e^{2\alpha r}\rangle^{-2}$ can be obtained using the Hellmann-Feynmann theorem (HFT) [27-29] taking into consideration the Hamiltonian H of a particular quantum system with a function of some parameter q, the energy eigenvalue $E_n(q)$ and the eigenfunction $\varphi_n(q)$ and it states that

$$\frac{\partial E_n(q)}{\partial q} = \langle\varphi_n(q)\left|\frac{\partial H(q)}{\partial q}\right|\varphi_n(q)\rangle \tag{26}$$

The effective Hamiltonian of the hyper radial function is given as

$$H = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu}\frac{(D+2l-1)(D+2l-3)}{4r^2} +$$

$$A + \frac{B}{(q+e^{2\alpha r})} + \frac{C}{(q+e^{2\alpha r})^2} + \frac{Fbe^{2\alpha r}}{(q+e^{2\alpha r})} + \frac{Gbe^{2\alpha r}}{(q+e^{2\alpha r})^2} \tag{27}$$

To obtain $\langle r^{-2}\rangle$, we let $q = l$

$$\frac{\partial E}{\partial l} = \langle \varphi_n(l) \left| \frac{\partial H(l)}{\partial l} \right| \varphi_n(l) \rangle \tag{28}$$

$$\frac{\partial E}{\partial L} = -\frac{4\alpha^2\hbar^2}{\mu}\times$$

$$\left\{ \frac{\frac{(n+\sigma)(2l+D-2)}{2\alpha^2 r_e^2 q}\left[\frac{d_2}{q}+d_1\right] + (\varphi-\beta-\gamma)\frac{d_2}{4\alpha^2 r_e^2 q^2}(2l+D-2)\left[\varphi+\beta+\frac{1}{4}-m\right]^{\frac{1}{2}}}{[2(n+\sigma)]^2} \right.$$

$$\left. -\frac{1(2l+D-2)}{16\alpha^2 r_e^2 q^2}\left[\varphi+\beta+\frac{1}{4}-m\right]^{-\frac{1}{2}} \right\} \left[\frac{(\varphi-\beta-\gamma)}{2(n+\sigma)}+\frac{(n+\sigma)}{2}\right] - \frac{d_0\hbar^2}{2\mu r_e^2}(2l+D-2) \tag{29}$$

Thus by the HFT, we have

$$\langle r^{-2}\rangle = \frac{2u}{\hbar^2(2l+D-2)}\left\{ -\frac{4\alpha^2\hbar^2}{\mu}\times \right.$$

$$\left[\frac{\frac{(n+\sigma)(2l+D-2)}{2\alpha^2 r_e^2 q}\left[\frac{d_2}{q}+d_1\right] + (\varphi-\beta-\gamma)\frac{d_2}{4\alpha^2 r_e^2 q^2}(2l+D-2)\left[\varphi+\beta+\frac{1}{4}-M\right]^{\frac{1}{2}}}{[2(n+\sigma)]^2} \right.$$

$$\left. -\frac{1(2l+D-2)}{16\alpha^2 r_e^2 q^2}\left[\varphi+\beta+\frac{1}{4}-M\right]^{-\frac{1}{2}} \right]$$

$$\left. \left[\frac{(\varphi-\beta-\gamma)}{2(n+\sigma)}+\frac{(n+\sigma)}{2}\right] - \frac{d_0\hbar^2}{2\mu r_e^2}(2l+D-2) \right\} \tag{30}$$

Similarly, by letting q=B, we obtain

$$\langle q+e^{2\alpha r}\rangle^{-1} = \frac{1}{q(n+\sigma)}\left[\frac{(\varphi-\beta-\gamma)}{2(n+\sigma)}+\frac{(n+\sigma)}{2}\right] \tag{31}$$

Similarly, by letting q=C, we obtain

$$\langle q+e^{2\alpha r}\rangle^{-2} = -\frac{4\alpha^2\hbar^2}{\mu}\left[\frac{-2\mu(n+\sigma)-\mu(\varphi-\beta-\gamma)(\varphi+\beta+\frac{1}{4}-M)^{\frac{1}{2}}}{8\alpha^2\hbar^2 q(n+\sigma)^2}\right.$$

$$+\frac{\mu(\varphi+\beta+\frac{1}{4}-M)^{-\frac{1}{2}}}{8\alpha^2\hbar^2 q(n+\sigma)^2}\Bigg]\left[\frac{(\varphi-\beta-\gamma)}{2(n+\sigma)}+\frac{(n+\sigma)}{2}\right] \tag{32}$$

4 Conclusions

In summary, we have investigated the D-dimensional schrödinger equation with multiparameter potential analytically for arbitrary lstate by means of the Nikiforov-Uvarov method and using a reasonable approximation. Expectation values $\langle r^{-2}\rangle$, $\langle q+e^{2\alpha r}\rangle^{-1}$ and $\langle q+e^{2\alpha r}\rangle^{-2}$ were calculated using the Hellmann-Feynman Theorem.

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