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INTEGRATION BY DIFFERENTIATION

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Dedicated to Prof. M.A. Pathan on his 75^{th} birth anniversary

Abstract: We employ an expression for the Laplace transform, based in integration by differentiation, to deduce the Post-Widders formula for the inversion of this transform. Besides, we apply the Kempf et al process to deduce the Lanczos generalized derivative.

Keywords and Phrases: Inversion of the Laplace transform, Post-Widders formula, Orthogonal derivative, Integration by differentiation, Lanczos derivative.

Mathematics Subject Classification: 44A10.

1. Introduction

If we know the Laplace transform [1]:

$$F(s) = \int_0^\infty e^{-st} f(t) dt, \tag{1}$$

the aim is to determine f(t); in [2] was obtained the following formula to do the integration in (1) via differentiation:

$$F(s) = f\left(-\frac{d}{ds}\right)\frac{1}{s} \tag{2}$$

In Sec. 2 we show that (2) leads to the Post-Widders expression [3-6] for the inverse Laplace transform. Kempf et al [2, 7] exhibit how to determine a definite integral via differentiation, in fact, they find the interesting expression:

$$\int_a^b F(x)dx = \lim_{t\to 0} F\left(\frac{d}{dt}\right)\left[\frac{e^{bt}-e^{at}}{t}\right] \tag{3}$$

The Sec. 3 contains an elementary proof of (3), besides we use (3) to obtain the Lanczos generalized derivative [8-14].

2. Inversion of the Laplace Transform

In (2) we apply the operator $\frac{d^m}{ds^m}$ and besides $f(t)$ is written in its Taylors series with respect to $t = b$:

$$F^{(m)}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f^{(n)}(b) \sum_{k=0}^{n} \binom{n}{k} b^{n-k} \left(\frac{d}{ds}\right)^{m+k} \frac{1}{s}$$

but $\left(\frac{d}{ds}\right)^{m+k} \frac{1}{s} = \frac{(-1)^{m+k}(m+k)!}{s^{m+k+1}}$ and we take $b = t$ with $s = \frac{m}{t}$, hence:

$$F^{(m)}\left(\frac{m}{t}\right) = \frac{(-1)^m t^{m+1}}{m^{m+1}} m! \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f^{(n)}(t) t^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k (m+k)!}{m! m^k}$$

$$= (-1)^m \left(\frac{t}{m}\right)^{m+1} m! \left[f(t) + t f^{(1)}(t) \frac{1}{m} + \right.$$

$$\left. \frac{t^2}{2!} f^{(2)}(t) \frac{1}{m}\left(1 + \frac{2}{m}\right) + \frac{t^3}{3!} f^{(3)}(t) \frac{1}{m^2}\left(5 + \frac{6}{m}\right) + \ldots \right],$$

therefore:

$$\lim_{m\to\infty} \frac{(-1)^m}{m!} \left(\frac{m}{t}\right)^{m+1} F^{(m)}\left(\frac{m}{t}\right) = f(t), \tag{4}$$

which coincides with the celebrated formula of Post-Widder [3-6] for the inversion of the Laplace transform.

3. Kempf et al Formula and Orthogonal Derivative

Here we show a simple manner to deduce (3):

$$\int_a^b x^n dx = \frac{1}{n+1}(b^{n+1} - a^{n+1}) = \left[\frac{d^n}{dt^n} \sum_{r=0}^{\infty} \frac{b^{r+1} - a^{r+1}}{(r+1)!} t^r \right]_{t=0},$$

$$= \lim_{t\to 0}\left[\frac{d^n}{dt^n}\frac{1}{t}\sum_{k=0}^{\infty}\frac{b^k-a^k}{k!}t^k\right] = \lim_{t\to 0}\frac{d^n}{dt^n}\frac{e^{bt}-e^{at}}{t},$$

then:

$$\int_a^b F(x)dx = \sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!}\int_a^b x^n dx = \lim_{t\to 0}\sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!}\frac{d^n}{dt^n}\frac{e^{bt}-e^{at}}{t},$$

hence (3) is immediate.

Now we apply (3) for the case $F(x) = xf(x+x_0)$, with $a = -b = -\epsilon$, therefore:

$$\frac{e^{bt}-e^{at}}{t} = 2\epsilon\left(1+\frac{\epsilon^2}{3!}t^2+\frac{\epsilon^4}{5!}t^4+...\right), \quad F(x) = f(x_0)x+f'(x_0)x^2+\frac{1}{2!}f''(x_0)x^3+...,$$

thus from (3):

$$\int_{-\epsilon} \epsilon x f(x+x_0)dx$$

$$= 2\epsilon\lim_{t\to 0}\left[f(x_0)\frac{d}{dt}+f'(x_0)\frac{d^2}{dt^2}+\frac{1}{2!}f''(x_0)\frac{d^3}{dt^3}+...\right]\left(1+\frac{\epsilon^2}{3!}t^2+\frac{\epsilon^4}{5!}t^4+...\right)$$

$$= 2\epsilon^3\left[\frac{1}{3}f'(x_0)+\frac{\epsilon^2}{5\cdot 3!}f'''(x_0)+\frac{\epsilon^4}{7\cdot 5!}f^{(5)}(x_0)+...\right],$$

then it is evident the expression:

$$f'(x_0) = \lim_{\epsilon\to 0}\frac{2}{\epsilon^3}\int_{-\epsilon}^{\epsilon} xf(x+x_0)dx, \tag{5}$$

which coincides with the Lanczos generalized derivative [8-14].

Remark. The relation (3) represents integration by differentiation, but (5) expresses the inverse process, that is, differentiation by integration.

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