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GROUP THEORETICAL ASPECTS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Dedicated to Prof. M.A. Pathan on his 75^{th} ***birth anniversary***

Abstract: In this survey article, the group theory of the 24 Kummer solutions of the Gauss second order ordinary differential equation and the group theory of the ${}_3F_2(a, b, c; d, e; 1)$ transformations of Weber-Erdelyi, giving rise to a new 72-element group.

1. Introduction

Leonhard Euler (1707 - 1783) is perhaps the first to study the hypergeometric functions, in 1748. The modern framework for the hyper geometric series and the corresponding hypergeometric functions is due to Gauss. Carl Friedrich Gauss (1777 - 1855), the German mathematician has been acknowledged as one of the three leading mathematicians of all time, with Archimedes (287 B.C. – 214/212 B.C.) and Sir Isaac Newton (1642 – 1727), being the other two. Besides his contribution to the theory of numbers, his outstanding work includes the discovery of the Method of Least Squares, the hypergeometric series and non-Euclidian geometry. His collected papers run to several volumes and were being edited at Göttingen in the 20th century. In 1812, recognizing the importance of the property of convergence of an infinite series, he published his comprehensive thesis on "*Disquisitiones generales circa seriem infinitam*" [1]. Historically, the geometric series:

$$(1-x)^{-1} \;=\; 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \;=\; \sum_{k=0}^{\infty} x^k,\ \forall\ 0 \le x < 1, \qquad (1)$$

the first theorem which one comes across in School as a special case of the Binomial theorem, when read as:

$$1+x+x^2+x^3+\cdots+x^n+\cdots = \sum_{k=0}^{\infty} x^k = (1-x)^{-1}, \ \forall\ 0 \le x < 1, \qquad (2)$$

is also the first summation theorem one comes across in life as a student of mathematics! Wallis [2], called a series with its nth term given by

$$a(a+b)(a+2b)\cdots(a+(n-1)b)$$

as the hypergeometric series. For $b=1$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad \text{with} \quad (a)_0 = 1, \qquad (3)$$

is called the Pochhammer symbol. Gauss [1] made a sweeping generalization and introduced to the world of mathematics the hypergeometric series, in 1812, as:

$$1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(+1)}\frac{z^2}{2!}+\cdots = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} = {}_2F_1(a,b;c;z), \qquad (4)$$

which takes one above or beyond ('hyper') the geometric series. For,
(i) when $a=1,\ b=c$, it is the geometric series; (ii) when $a=b=1,\ c=2,\ z=1$ it is the harmonic series:

$${}_2F_1(1,1;2;1) = \sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(2)_n}\frac{1^n}{n!}. \qquad (5)$$

From the ratio test, it follows that the series converges for all $|z|<1$, diverges for all $|z|>1$ and for $|z|=1$, converges if $c-a-b>0$.

It is to be emphasized that Gauss (1812), while announcing the hypergeometric series stated explicitly that it should not be considered as a series in one variable z, and a, b treated as numerator parameters and c as the denominator parameter, but that it should be treated as a function of four variables $F(a,b,c,z)$. It was also shown to be the solution of the second ordinary differential equation characterized by three regular singular points at $0,\ 1,\ \infty$:

$$z(1-z)\frac{d^2u(z)}{dz^2}+[c-(a+b+1)z]\frac{du(z)}{dz}-abu(z) = 0, \qquad (6)$$

where a, b, and c, are complex parameters, and has one solution as the hypergeometric series ${}_2F_1(a,b;\ c;\ z)$, which belongs to a set of 24 functions. Kummer (1810

- 1893) published a set of 6 distinct solutions of the hypergeometric equation. Each of these six solutions has four forms, related to one another by Euler's transformations, giving 24 forms in total [3], given in the classical monographs of Bailey [4] and Lucy Slater [5]. That these 24 solutions are related to one another by a finite group of transformations was observed by the author, with Leuvens and vander Jeugt [6]. This finite group of order 24 (or, by natural extension, when the trivial numerator parameter transformations of a, b makes it a group of order 48) was characterized. The finite group of 24 Kummer solutions is shown to be isomorphic to the symmetric group S_4. The idea was to find a mapping of the four 'variables' of the Gauss function onto the six parameters that may be associated with the six sides of the ordinary cube. If the six variables of the cube are $x_i, (i = 1, \ldots 6)$ satisfying the constraint

$$\sum_{0}^{\infty} x_i = 0 \tag{7}$$

and the function is :

$$f(x) \; = \; F(\frac{1}{2} + x_1 + x_2 + x_3, \; \frac{1}{2} + x_1 + x_2 + x_4; \; 1 + x_1 - x_6; \; -\frac{x_1 + x_6}{x_3 + x_4}). \tag{8}$$

Identifying the four arguments of F with $a,\ b,\ c$ and z; solving this system with respect to the x_i leaves one degree of freedom (since there is constraint on the six parameters associated with the sides of the cube, and so only 5 of the x_i's are independent.).

Consider any element of g of the group, the action of g on x is determined by permuting the indices of the x_i. So, acting with $g_1 = (2, 4, 5, 3)$ on $f(x)$ gives

$$f(g_1 \cdot x) \; = \; F(\frac{1}{2} + x_1 + x_2 + x_4, \; \frac{1}{2} + x_1 + x_4 + x_5; \; 1 + x_1 - x_5; \; -\frac{x_1 + x_6}{x_2 + x_5}) \tag{9}$$

and this is equal to $F(b,\ c - a,\ ;\ ;\ z/(z-1))$ when the original $f(x)$ is identified with $F(a,\ b, ;\ c;\ z)$. Similarly, one finds with $g_2 = (1, 2, 6, 5)$ that

$$f(g_2 \cdot x) \; = \; F(1 + a - c, \; 1 + b - c; \; 1 + a + b - c; \; 1 - z). \tag{10}$$

For each element g of this group, the corresponding function $f(g \cdot x)$ is given in a table in that article in which every elements of the group is associated with one of the 24 Kummer solutions, thereby establishing a unique one-to-one correspondence between the 24 symmetries of the cube and the 24 solutions of Kummer. In fine, the two pages of output in any of the texts listing the 24 Kummer solutions for the Gauss ODE, is now a one line statement, with the action of any element of our group is obtained by the 24 permutations of the indices, as follows:

$$g \;\; f(x) \rightarrow \frac{C(g \cdot x)}{C(x)} \; f(g \cdot x), \tag{11}$$

where

$$C(x) \;=\; (-1)^{\alpha}(x_1 + x_6)^{\beta}(x_2 + x_5)^{\gamma}(-x_3 - x_4)^{\delta}, \tag{12}$$

with

$$\alpha \;=\; \frac{x_1 - x_6}{4} + \frac{1}{12}; \qquad \beta \;=\; \frac{x_1 - x_6}{2} + \frac{1}{6}$$

$$\gamma \;=\; \frac{(x_2 - x_5)}{2} + \frac{1}{6}; \qquad \delta \;=\; \frac{x_1 - x_6}{2} + \frac{x_2 - x_5}{2} + \frac{1}{3}.$$

We extended the group to a group of order 48, by adding the additional generator $g_3 \;=\; (3, 4)$ to g_1, g_2 corresponding to a reflection about a plane.

A Group of ${}_3F_2(1)$ transformations

A recursive use of the transformation for a terminating ${}_3F_2(1)$ series used by Weber and Erdelyi (1952), led them to obtain a second transformation from a given transformation. It has been shown by K. Srinivasa Rao et. al (1992) that they belong to a 72 element group associated with 18 terminating series. The generators, conjugacy classes, invariant subgroups, characters and dimensions of irreducible representations for this group were presented.

The transformation for a terminating ${}_3F_2(1)$ proposed by Weber and Erdelyi (1952) :

$${}_3F_2\begin{pmatrix} a, b, -N \\ d, e \end{pmatrix} = \frac{\Gamma(d, d + N - a)}{\Gamma(d + N, d - a)} {}_3F_2\begin{pmatrix} a, e - b, -N \\ 1 + a - d - N, e \end{pmatrix}. \tag{13}$$

This formula is one of a set (cf. Bailey 1935) obtained by Whipple (1925). If the five parameters of the ${}_3F_2$ on the l.h.s. of (13) are denoted by the column vector :

$$\vec{x} = (a, b, 1 - N, d, e)\,, \tag{14}$$

then the parameters of the ${}_3F_2$ on the r.h.s. of (13) are obtained when the matrix :

$$g_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

operates on $\vec{x}$. Note that $1-N$ is used instead of $-N$, as a component of the column vector $\vec{x}$, since it represents the number of terms in a terminating series. However, ${}_3F_2(a,b,-N;d,e)$ will be denoted by ${}_3F_2(\vec{x})$.

Using (13) again, with the roles of d and e interchanged, to transform the r.h.s. of (13), Weber and Erdelyi obtained the transformation :

$$ {}_3F_2\begin{pmatrix} a,b,-N \\ d,e \end{pmatrix} = \frac{\Gamma(d,e,e+N-a,d+N-a)}{\Gamma(d+N,e+N,d-a,e-a)}\,{}_3F_2\begin{pmatrix} a,1-s,-N \\ 1-b+d-s,1-b+e-s \end{pmatrix}, \tag{15} $$

where $s=d+e-a-b+N$. The question arises as to whether this recursive use of the Weber-Erdelyi transformation (12) can be continued. In fact, such a procedure when continued results in a group of 72 transformations, which are the 18 terminating ${}_3F_2$ series (see Appendix of KSR, et. al. 1992) on which are superposed the $a \leftrightarrow b$, $d \leftrightarrow e$ interchanges.

Let g_2 and g_3 be the matrices :

$$ \begin{bmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{bmatrix} \tag{16} $$

which interchanges a and b when it operates on $\vec{x}$ and interchanges d and e when it operates on $\vec{x}$, respectively. By forming all possible products of all possible powers of g_1, g_2 and g_3, a group of 72 transformation matrices can be generated which provides a 5×5 representation for the terminating series, with (14) as the basis. Thus, g_1, g_2 and g_3 are the generators of a group G_T for the transformations of a terminating ${}_3F_2$ series, with $g_i^2=\mathbf{1}$, for $i=1,2,3$.

A similarity transformation, $u^{-1}g_iu$, with :

$$ u=\begin{bmatrix} 1&0&1&0&0\\ 0&1&1&0&0\\ 0&0&3&0&0\\ 0&0&2&1&0\\ 0&0&2&0&1 \end{bmatrix} \quad \text{and} \quad u^{-1}=\frac{1}{3}\begin{bmatrix} 3&0&-1&0&0\\ 0&3&-1&0&0\\ 0&0&1&0&0\\ 0&0&-2&3&0\\ 0&0&-2&0&3 \end{bmatrix}, \tag{17} $$

block diagonalizes the generators, and hence all the $g\in G_T$, thereby reducing the generators for the 5×5 representation into the generators for a one-dimensional identity irrep (due to $-N$ being kept fixed in (13)) and the generators for a fourdimensional faithful irrep given by :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{18}$$

The results obtained by Srinivasa Rao et.al.(1992) in their definitive study of the group theoretical basis for hte terminating ${}_3F_2(1)$ series are summarized below:

Two elements h and h' of a group G are said to be conjugate if there exists a $g\varepsilon G$ such that $h' = ghg^{-1}$. This defines an equivalence relation on G, the equivalence classes being called the conjugacy classes. Analysis of G_T reveals that there are 9 conjugacy classes $K_1, \ldots K_9$.

Following the general theory of group representations (ref. Wybourne 1970, or Messiah 1964), the table of characters of the irreducible representations (irreps) of G_T has been obtained. As there are 9 conjugate classes, there are 9 inequivalent irreps which are denoted by $D^{(1)}, \ldots D^{(9)}$. Four irreps are of dimension 1, one is of dimension 2 and four are of dimension 4. It is only the 4-dimensional irreps which are faithful.

All the invariant subgroups H of G_T have been found by Srinivasa Rao et.al. Among these are the proper abelian invariant subgroups, hence G_T is neither simple nor semi-simple. By definition, a subgroup H is a self-conjugate or normal divisor or invariant subgroup iff $G_T H G_T^{(-1)} = h$. To find the invariant subgroups of G_T, one can form unions of conjugacy classes and check if they close under the group multiplication law. The following inclusion table gives a complete list of the invariant subgroups of G_T, with a subscript of H denoting the order of that subgroup:

$$H_9 \subset H_{18} \begin{cases} \subset H_{36} \subset G_T \,, \\ \subset H'_{36} \subset G_T \,, \\ \subset H''_{36} \subset G_T \,, \end{cases} \tag{19}$$

where

$$\begin{aligned} H_9 &= K_1 \cup K_2 \cup K_3 \,, \\ H_{18} &= H_9 \cup K_6 \,, \\ H_{36} &= H_{18} \cup K_9 \,, \\ H'_{36} &= H_{18} \cup K_4 \cup K_7 \,, \\ H''_{36} &= H_{18} \cup K_5 \cup K_8 \,. \end{aligned} \tag{20}$$

It should be noted that, in terms of the three generators g_i (or g'_i) introduced previously, one can write

$$K_6 = g_2 g_3 H_9\,, \quad K_9 = g_1 g_2 H_{18}\,, \quad K_4 \cup K_7 = g_1 H_{18}\,, \quad K_5 \cup K_8 = g_2 H_{18}\,, \tag{21}$$

such that the invariant subgroups (23) can be characterized as follows in terms of H_9 and the three generators :

$$\begin{aligned} H_{18} &= H_9 \cup g_2 g_3 H_9\,, \\ H_{36} &= H_9 \cup g_2 g_3 H_9 \cup g_1 g_2 H_9 \cup g_1 g_3 H_9\,, \\ H'_{36} &= H_9 \cup g_2 g_3 H_9 \cup g_1 H_9 \cup g_1 g_2 g_3 H_9\,, \\ H''_{36} &= H_9 \cup g_2 g_3 H_9 \cup g_2 H_9 \cup g_3 H_9\,. \end{aligned} \tag{22}$$

The smallest invariant subgroup, H_9, is easy to characterize. In fact $H_9 = C_3 \times C_3$, the direct product of two cyclic groups on three elements. In terms of the Whipple parametrization, the generators of the two C_3's are (012) and (345). It is now obvious that H_9 is an abelian invariant subgroup of G_T.

It should be noticed that all the invariant subgroups of G_T can be found using the character table and the fact that those elements h of G_T with $\phi(h) = \phi(\mathbf{1})$, where ϕ is a (not necessarily simple) character of G_T, form an invariant subgroup (Ledermann 1977, Theorem 2.7).

Conversely, having the list of all invariant subgroups of G_T, one can reconstruct the character table. Indeed, the first character $\chi^{(1)}$ is trivial. Next, if N is one of H_{36}, H'_{36} or H''_{36}, G/N is the 2 element group C_2, with non-trivial simple character $(1,-1)$. Using the "lifting process" (Ledermann 1977, Theorem 2.6), one obtains the simple characters $\chi^{(2)}$, $\chi^{(3)}$ and $\chi^{(4)}$ from H''_{36}, H'_{36} and H_{36} respectively. This completes the list of simple characters with $\chi_1^{(i)} = 1$. In order to find the remaining simple characters, the theory of induced characters can be used. If H is a subgroup of G for which a character ${}^H\phi$ is known, then

$${}^G\phi_i = \frac{m}{k_i} \sum_w {}^H\phi(w)\,, \qquad w \in K_i \cap H \tag{23}$$

is a character (simple or compound) of G. Herein, m is the index of H and k_i is the order of K_i. As the simple characters of an abelian group are well known, H is chosen to be $H_9 = C_3 \times C_3$, thus $m = 72/9 = 8$. Using the trivial character of H, ${}^H\phi^{(1)} = (1,1,1,1,1,1,1,1,1)$, one finds ${}^G\phi^{(1)} = (8,8,8,0,0,0,0,0,0)$. By means of the inner product for characters of G_T,

$$\langle \phi | \psi \rangle = \frac{1}{72} \sum_{i=1}^{9} k_i \phi_i \psi_i\,, \tag{24}$$

it is found that

$$\langle {}^G\phi^{(1)}|\chi^{(1)}\rangle = \langle {}^G\phi^{(1)}|\chi^{(2)}\rangle = \langle {}^G\phi^{(1)}|\chi^{(3)}\rangle = \langle {}^G\phi^{(1)}|\chi^{(4)}\rangle = 1. \tag{25}$$

Thus, subtracting $\chi^{(1)}, \ldots, \chi^{(4)}$ from ${}^G\phi^{(1)}$, one obtains

$${}^G\phi' = (4, 4, 4, 0, 0, -4, 0, 0, 0). \tag{26}$$

Since all one-dimensional irreps have been found and

$$\langle {}^G\phi'|{}^G\phi'\rangle = 4, \; \Rightarrow^G \phi' \tag{27}$$

is twice a simple character, i.e. ${}^G\phi' = 2\chi^{(5)}$. The next simple character, $\chi^{(6)}$, is immediately deduced from our defining representation for $g_1, \ g_2, \ g_3$. Using a non-trivial character of H,

$${}^H\phi^{(2)} \;=\; (1, 1, 1, \omega, \omega, \omega, \omega^2, \omega^2, \omega^2), \tag{28}$$

where

$$\omega^2 \;+\; \omega \;+\; 1 \;=\; 0, \tag{29}$$

the inducing process leads to ${}^G\phi^{(2)} = (8, 2, -4, 0, 0, 0, 0, 0, 0)$. One can verify that the inner product of ${}^G\phi^{(2)}$ with $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$, $\chi^{(4)}$ and $\chi^{(5)}$ is zero, and that

$$\langle {}^G\phi^{(2)}|\chi^{(6)}\rangle = 1. \tag{30}$$

Subtracting $\chi^{(6)}$ from ${}^G\phi^{(2)}$, one obtains

$${}^G\phi'' = (4, 1, -2, 0, -2, 0, 0, 1, 0). \tag{31}$$

Since

$$\langle {}^G\phi''|{}^G\phi''\rangle = 1, \tag{32}$$

it is a simple character, i.e.

$${}^G\phi'' = \chi^{(7)}. \tag{33}$$

Two more simple characters $\chi^{(8)}$ and $\chi^{(9)}$ need to be found. Using the orthogonality property satisfied by the columns of the character table of G_T, namely

$$\sum_{l=1}^{9} \chi_i^{(l)}\chi_j^{(l)} = \frac{72}{k_i}\delta_{ij}\,, \tag{34}$$

it is a straightforward exercise to complete the character table.

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