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FOURIER ANALYSIS TO WAVELET ANALYSIS

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Dedicated to Prof. M.A. Pathan on his 75^{th} ***birth anniversary***

Abstract: From a very modest presentation as an introductory composition of wavelets by Chui in 1992 to a very specialist and advanced monographs by Meyer in 1990, and by Daubechies in 1992, one will certainly experience the beauty of this subject, which in the recent time has attracted both the pure and applied mathematicians. *Wavelet transform*, more correctly called the *integral wavelet transform*, is one of the two entities of the wavelet analysis. Possibly the *window Fourier transform*, also called the Gabor transform (first introduced by Gabor in 1946), is the initiation for wavelet transform. In this brief note we attempt to discuss some of its aspects.

Keywords: Wavelet transform, Fourier transform, Gabor transform, Multiresolution analysis, Wavelet series, Fourier series, General distributions, Tempered distributions.

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1. Introduction

Wavelets stand at the intersection of the frontiers of mathematics, scientific computing and signal and image processing. It has been one of the major research direction in science in the last decade and is still undergoing rapid growth. Wavelet is a versatile tool in very aspect of mathematical context and possesses great potential for applications owing to viewing it as a new basis for *representing functions*. Some consider it as a technique for time frequency analysis and others think of it as a new mathematical subject. Wavelet analysis provides another fascinating interface between physics and mathematics. It were more instrumental in the explosive growth of the subject than were mathematical physicist.

In this paper, the brief historical survey about this branch of analysis is introduced through some monographs and articles, which raises the awareness among researchers in this field to make others feel involved. The paper concludes with an application, which involves wavelet transform, fractional calculus and a special function space.

2. Historical Overview

Fourier analysis is an established subject in the core of pure and applied mathematical analysis. Not only the techniques in this subject are of fundamental importance in all areas of science and technology, but both **integral Fourier transform** and **Fourier series** also have significant physical interpretations . In addition, the computational aspects of Fourier series are especially attractive, mainly because of the orthogonality property of the series and owing to its simple expression in terms of only two functions $\sin x$ and $\cos x$. It is asserted that any 2π periodic function $f(x)$ is the sum

$$a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

of its Fourier series, a_0, a_k, b_k being its coefficients.

Given a signal, say a sound or an image, Fourier analysis easily calculates the frequencies and the amplitudes of those frequencies, which make up a the signal. This provides a broad overview of the characteristics of the signal, which is important for theoretical considerations. Although Fourier inversion is possible under certain circumstances, Fourier methods are not always a good tool to recapture the signal, particularly if it is highly non-smooth : too much Fourier information is needed to reconstruct the signal locally. In such cases, wavelet analysis is often very effective, because it provides a simple approach for dealing with the local aspects of a signal. **Wavelet analysis** provides new methods for removing noise from signals, that complement the classical methods of Fourier analysis.

Wavelets, developed mostly during the last 25 years, is connected to older ideas in many others fields, including pure and applied mathematics and engineering. The concept of wavelets or **ondelettes** started to appear in the literature only in 1980's. This new concept can be viewed as a syntheses of various ideas which originated from different disciplines, including mathematics (Calderón Zygmund operators and Littlewood - Paley theory), physics (coherent states formalism in quantum mechanics and in renormalization group) and engineering (quadratic mirror filters, side band coding is signal processing and pyramidal algorithm in image processing).

In early forties, those who used the Fourier approach in order to analyze natural behaviours, were frustrated with the limitations of Fourier transform and Fourier series in the investigation of physical phenomena with non-periodic behaviour and local variations, which possibly raised the need for simultaneous time frequency of **Gabor's short time Fourier transform** in 1946, and so called **Wigner-Ville transform** in 1947. But the common ingredient of these two transforms is the sinusoidal kernel in the core of their definitions, so that both high and low frequency behaviours are investigated in the same manner and any signal under investigations is matched by the same rigid sinusoidal waveform. In place of the sinusoidal kernel as modulation (for phase shift), a French geophysicist, Morlet [30, 31], introduced the operation of dilation, while keeping the translation operations, developed an algorithm for the recovery of the signals under investigation from this wavelet transform. Then, a French theoretical physicists, Alex Grossman, who quickly recognized the importance of the Morlet wavelet transform, which is something similar to coherent states formalism in quantum mechanics, developed an exact inversion formula for the wavelet transform. Then, by the joint venture of mathematical physics group in Marseille, led by Grossman, in collaboration with Daubechies, Paul and others, extended Morlet's discrete version of wavelet transform to the continuous version, by relating it to the theory of coherent states in quantum physics. This was the notion of the integral (or continuous) wavelet transform.

In order to eliminate the above said weakness of the Fourier analysis, Dennis Gabor [14], a Hungarian British physicist and engineer, first introduced the **Windowed Fourier transform (or the short-time Fourier transform,** or more appropriately the **Gabor transform**) by using a Gaussian distribution function as the window function. The idea of using a window function lies in order to localize the Fourier transform and then shift the window to another position, and so on.

The remarkable feature of the Gabor transform is the local aspect of the Fourier analysis, with the time resolution equal to the size of the window. In fact, it deals with discrete set of coefficients which allows efficient numerical computation of those coefficients. However, the Gabor wavelets suffers from some serious algorithmic handicaps and shortcomings which have, successfully, been solved by Henrique Malvar [23, 24]. Malvar wavelets are much more effective and superior to other wavelets, including Gabor wavelets and Morlet-Grossman wavelets.

The development of the wavelet transform and mathematical analysis of the wavelet transform had really not begun, until a year later in 1985, when Meyer, learnt about the work of Morlet and the Marseille group, recongnized immediately the deep connection of Morlet's algorithm to the notion of resolution of identity in harmonic analysis due to Calderón in 1964. He then applied the Littlewood-Paley theory to the study of wavelet decomposition. In this regard, Yves Meyer may be considered as the founder of this mathematical subject, **which we call wavelet analysis**.

Since wavelet analysis is built on Fourier analysis, Meyer's book [26] devotes a brief discussion on distributions, the Poisson summation formula, Shannon's sampling theorem and the Littlewood-Paley theory. He also explains the construction of wavelets and the application of wavelet series representations to the analysis of the most important function spaces, such as Hölder, Hardy, Block and Besov and also the notion of holomorphic wavelets.

The next great achievement of wavelet analysis was due to Daubechies et al. [10] which suggests a new construction of painless non -orthogonal wavelet expansion. During 1985-86, further work of Lemarié and Meyer [18] on the first construction of a smooth orthonormal basis on $\mathbb{R}$ and $\mathbb{R}^N$, marked the beginning of their famous contributions to the wavelet theory. The collaborations of Meyer and Mallat, culminated with the remarkable discovery by Mallat of new formalism [21, 22], came to be known as **multiresolution analysis**.

Inspired by the work of Meyer and Daubechies [9] made a remarkable contribution to wavelet theory by constructing families of **compactly supported orthonormal wavelets** with some degree of smoothness. But after a great success, she reconginzed that it is difficult to construct wavelets that are symmetric, orthogonal and compactly supported. Chui and Wang [5, 6] introduced **compactly spline wavelets,** and **semi-orthogonal wavelet** analysis. As a natural extension of wavelet analysis, Coifman et al. [7, 8] discovered wavelet packets which can be used to design efficient schemes for the representation and compression of acoustic signals and images.

3. From Fourier Analysis to Wavelet Analysis

Fourier analysis usually refers to **(integral) Fourier transform** and **Fourier series**. A Fourier transform is the Fourier integral of some function f defined on the real line $\mathbb{R}$. When f is thought of as an analog signal, then its domain of definition $\mathbb{R}$ is called **continuous time domain**. In this case, the Fourier transform $\widetilde{f}$ of f describes the spectral behaviour of the signal f . Since the spectral information is given in terms of frequency, the domain of definition of Fourier transform f , which is again $\mathbb{R}$, is called the **frequency domain**. On the other hand, a Fourier series is a transformation of **bi-infinite sequences** to periodic functions. Hence, when a bi-infinite sequence is thought of as a digital signal, its domain of definition, which is the set $\mathbb{Z}$ of integers, is called the **discrete time domain**. In case of its Fourier series, again describes the spectral behaviour of digital signal, and the domain of a Fourier series is again the real line $\mathbb{R}$ which is the frequency domain. However, since Fourier series are 2π periodic, the frequency domain $\mathbb{R}$ in this situation, is usually identified with the unit circle.

The importance of both the Fourier transform and the Fourier series not only form the significance of their physical interpretations such as frequency analysis of signals, but also claim the fact, that Fourier analysis techniques are extremely powerful, for instance, in the study of wavelets analysis, the Poisson summation functions, and Parseval identities for both series and the integrals; Fourier transforms of the Gaussian, convolution of functions and delta distributions etceteras are often encountered.

Analogous to Fourier analysis, there are two important mathematical entities in **wavelet analysis,** the **integral wavelet transform** and **wavelet series.** The integral wavelet transform is defined to be the convolution with respect to the dilation of the reflection of some function $\widetilde{\psi}$ called a **basic wavelet,** while the wavelet series is expressed in terms of a single function ψ, called an R**-wavelet** (or simply a **wavelet**) by means of two very simple operations : **binary dilations** and **integral translations**. But unlike, Fourier analysis, since both are continuous and discrete and wavelet transform are defined on the real line group, these two components are intimately related. For instance, two functions ψ and $\widetilde{\psi}$ in $L^2(-\infty,\infty)$ constitute a pair of **dual wavelets**, if two families $\{\psi_{j,k}\}$ and $\{\widetilde{\psi}_{j,k}\}$ where j and k run over the set of all integers, are biorthogonal Riesz bases of $L^2(-\infty,\infty)$. Here for every function f defined on the real line, the notation $f_{j,k} = 2^{j/2} f(2^j - k)$ has been used. Hence, the relation between the continuous and discrete wavelet transform is evident from the observation that, any f in $L^2(-\infty,\infty)$ the coefficients (which constitute the discrete wavelet transform of f) of the series expansion of f in terms of the Riesz basis $\{\psi_{j,k}\}$ are the values of the continuous wavelet transform of f with the dual wavelet $\widetilde{\psi}$ as the convolution kernel (or analyzing wavelets) evaluated at the time scale positions $(k2^{-j}, 2^{-j})$. When some appropriate frequency ω_0 of a single function f has been identified, the change of scales (say by 2^{-j} for some integer j) reveals the frequency context at $2^j\omega_0$ of the signal, with known location near $k2^j$ in time (or spatial) axis. Furthermore, since the width of $\widetilde{\psi}_{j,k}$ narrows or widens as j increases or decreases, the wavelet transform has the so called **zoom in** and **zoom out** capabilities. This is one of the main reason that wavelet analysis is very useful for time frequency analysis.

4. Gabor transform and their basic properties

Time frequency analysis has always been challenging in signal processing. In the study of signals, represented by a function $f(t)$ in the multidimensional case, $f(t)$ represents an image or a video signal , the idea of frequency analysis can only be local in time.

The Fourier transform of a signal of a function $f$is usually defined by[9]

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-iwt}dt \tag{1}$$

One of the remarkable features of this transform is, that the integration of the signal is performed over the whole real line $\mathbb{R}$ so that every point of $\mathbb{R}$ contributes to the analysis of $\widehat{f}(w)$. The inversion (or reconstruction) formula is given by .

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)e^{iwt}dw \ . \tag{2}$$

In case of the frequency-modulated signals, the idea of local frequency plays an important role and hence, the study of **local Fourier transforms** becomes important.

In order to determine the local information in the Fourier transform analysis, it is necessary to use an analyzing function ψ which has localization properties, both in frequency (around some mean frequency (w_ψ)) and in time (around some mean time t_ψ) domains. Such a function ψ is called a **wavelet** which must be *wave of finite duration.* In view of the Parseval relation for the Fourier transform

$$(f, \psi) = (\widehat{f} \ , \widehat{\psi}) \tag{3}$$

where $f, \psi \in L^2(\mathbb{R})$ and (f, ψ) is the inner product in $L^2(\mathbb{R})$, physically, (f, ψ) can be treated as the average information of f in the vicinity of t_ψ and $(\widehat{f} \ , \widehat{\psi})$ as the average information of $\widehat{f}$ in the neighbourhood of w_ψ .

In order to incorporate both time and frequency localization properties in one single transform function, Dennis Gabor introduced the windowed Fourier transform (or the Gabor transform) by using a Gaussian distribution function as a window function $g_a(t-b)$, where a measures the width of the window, and the parameter b is used to translate the window in order to cover the whole time domain. The remarkable property of the Gabor transform is the local aspect of Fourier transform with time resolution equal to the size of the window. Using the canonical coherent state, the *continuous Gabor transform* (*Windowed Fourier transform*) of f with respect to g , denoted by $\widetilde{f}_g(\upsilon, t)$, is defined by

$$\mathcal{G}[f](\upsilon, t) = \widetilde{f}_g(\upsilon, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\overline{g}(\tau - t)e^{i\tau t}d\tau$$

$$= \frac{1}{\sqrt{2\pi}}(f, \overline{g}_{\upsilon,t}) \qquad , \tag{4}$$

where $f, g \in L^2(\mathbb{R})$ with the inner product (f, g). Clearly, the Gabor transform $\widetilde{f}_g(\upsilon, t)$, of a given signal f , depends on both , the frequency υ and time t.

In practical applications, f and g represent signals with finite energy. In quantum mechanics, $\widetilde{f}_g(w, t)$ is referred to as the canonical coherent state representation of f and used by Glauber in quantum optics. The continuous Gabor transform defines the properties as linearity, translation, modulation, conjugation, and Parseval formula. Considering the Paresval identity, the Gabor transformation is an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. The **discrete Gabor transform** is defined by

$$F(m, n) = \int_{-\infty}^{\infty} f(t)\overline{g}_{m,n}(t)dt = (f, g_{m,n}) \quad , \tag{5}$$

the double series $\sum\limits_{m,n=-\infty}^{\infty} F(m, n)g_{m,n}(t)$ is called the **Gabor series** of $f(t)$.

In many applications, it is more convenient to learn from a numerical point of view, to deal with the discrete transforms rather than continuous ones. The theory of Gabor transform has been generalized by Janseen [15, 16] for tempered distributions. Among many important results, he proved that any tempered distribution can be written as

$$\sum_{m,n} c_{m,n}\ g_{m,n}(t) \quad ,$$

where g is not necessarily a Gaussian function, but a function $g \in S$, where S is the **Schwartz space of generalized function** .The n-dimensional Gabor transform can also be defined as

$$\mathcal{G}[f](\upsilon, t) = \widetilde{f}_g(\upsilon, t) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x)g(x-t)e^{i(\upsilon,x)}dx \ , \tag{6}$$

where $\upsilon, t, x \in \mathbb{R}^n, x = (x_1, \ldots, x_n)$ and $x \cdot \upsilon = x_1\upsilon_1 + x_2\upsilon_2 + \cdots + x_n\upsilon_n$. The Gabor transformation provides a representation of n-variables by a function $2n$ variables.The Gabor transform has been found to be very useful in many physical and engineering applications, including signal processing and quantum optics.

5. Continuous Wavelet Transforms and their Basic Properties

Morlet [30, 31] modified the Gabor wavelets to study the layering of sediments in a geophysical problem of oil exploration, which led to the discovery of the wavelet transform, seems to be an efficient and effective time frequency representation algorithm. The major difference between the Morlet wavelet representation and the Gabor wavelet, is that the former has a more and more acute spatial resolution as the frequency gets higher and higher.

Based on the idea of wavelets as a family of functions, constructed from translation and dilation of a single function ψ, called the **mother wavelet**, is define by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right) \quad , a, b \in \mathbb{R}, a \neq 0 \tag{7}$$

where a is called the **scaling parameter**, which measures the degree of compression or scale, and b as **translation parameter**, which determines the time location of the wavelet. If $|a| < 1$, the above equation is the compressed version of the mother wavelet and corresponds to mainly higher frequencies.

The success of Morlet numerical algorithms prompted Grossman to make a more extensive study of the Morlet wavelet transform which led to the recognition, that wavelets $\psi_{a,b}(t)$ corresponds to a square integrable representations of the affine group.

Definition 1 (Wavelet) : A wavelet is a *function* $\psi \in L^2(\mathbb{R})$ which satisfies the condition

$$\int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}(w)\right|^2}{|w|} dw < \infty \tag{8}$$

where $\widehat{\psi}(w)$ is the Fourier transform of $\psi(t)$.

Definition 2 : (Continuous Wavelet Transform) : If $\psi \in L^2(\mathbb{R})$, and $\psi_{a,b}(t)$ is given by (7), then the integral transformation W_ψ, defined on $L^2(\mathbb{R})$, given by

$$W_\psi[f](a,b) = (f, \psi_{a.b}) = \int_{-\infty}^{\infty} f(t)\overline{\psi_{a,b}(t)}dt \quad , \tag{9}$$

is called a *continuous wavelet transform* of $f(t)$. The continuous wavelet transform also have basic properties as linearity, translation, dilation, symmetry , parity, antilinearity and Parseval relation.

6. Multiresolutiuon Analysis and Construction of Wavelets

There arise some difficulties in dealing with frames due to lack of orthogonality. Haar wavelet, which is defined by

$$\psi(t) = \begin{cases} 1 & , \ 0 \leq t < 1/2 \\ -1 & , \ 1/2 \leq t < 1 \\ 0 & , \quad \text{otherwise} \end{cases} \quad ,$$

forms an orthogonal system of wavelets based on the so called, **multiresolution analysis**. This is the formal approach to construct orthonormal wavelet bases, using a definite set of rules and procedures. In application, it is an effective mathematical framework for hierarchical decomposition of a signal or an image into components of different scales.

7. Recent Developments

The recent developments and outcomes are exponential. Walter defines wavelets and generalized functions [35] and in another publication [36], he introduces wavelets on $\mathbb{R}$ and analytic and harmonic wavelets in half planes; where he proves that the series of the said nomenclatures can be used to define the analytic representation of some tempered distributions. Debnath [12], introduces the wavelet transform and Gabor transform and described their application. This paper deals with brief historical introduction about wavelets and is self explanatory. Debnath and Mikusiński [13], introduced the Hilbert space and their application to wavelets. Pathak [33] investigated a self adjoint method and a complex inversion formula for the wavelet transform and further, the wavelet transform of generalized function is also discussed. Chui and Li [4] defines the notion of functional wavelet transform using the duals as analyzing wavelets, which retains some of the properties of the integral wavelet transform of Grossman and Morlet, such as the property of vanishing moments. Jiang [17], considered the wavelet transform associated to the Weyl-Poincare group and its quotient group, and then gave an orthogonal decomposition of L^2- space on the cartan domain. Using the concept of Fréchet space of distribution, Pandey [32] defines weighted modulation spaces on a locally compact abelian group, and proves a theorem on their wavelet representation. In [25] the wavelet series characterization of various classes of tempered distributions is presented which consists of derivatives of $L_p, p \geq 1$ functions. Bielecki et al. [2] considered the signals as tempered distribution which is used for deriving a multiresolution analysis of spaces of signals. They also introduce **Wavelet-Stieltjes transforms** and prove uniqueness theorem for it and compare wavelet transform with them.

8. Wavelet Transform of Fractional Integrals for Integrable Boehmians [20]

In what follow is an excellent combination of three most powerful entity of applicable analysis, which is not found to have appeared before. This deals with the wavelet transform of fractional integral operator (the Riemann-Liouville operators) on Boehmian spaces [20] . *By virtue of the existing relation between the wavelet transform and the Fourier transform, we obtained integrable Boehmians defined on the Boehmian space for the wavelet transform of fractional integrals.*

8.1 Brief Description

Definition 3 [34, p.33] : Let $\varphi(x) \in L_1(a,b)$. Then the integrals

$$(I^{\alpha}_{a+}\varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\varphi(t)dt \ , \ x > a \ , \tag{10}$$

$$(I^{\alpha}_{b-}\varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1}\varphi(t)dt \ , \ x < b \ , \tag{11}$$

where $\alpha > 0$, are Riemann-Liouville fractional integrals of order α . They are also known as *left-sided* and *right-sided fractional integrals*, respectively. Indeed, these integrals are extensions from the case of a finite interval $[a,b]$ to the case of a half-axis, given by

$$(I^{\alpha}_{0+}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\varphi(t)dt \quad , \quad 0 < x < \infty \tag{12}$$

while for the whole axis, it is given, respectively, by [34, p. 94]

$$(I^{\alpha}_{+}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1}\varphi(t)dt \quad , \quad -\infty < x < \infty \tag{13}$$

and

$$(I^{\alpha}_{-}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1}\varphi(t)dt \quad , \quad -\infty < x < \infty \tag{14}$$

The Fourier transform of the fractional integrals $I^{\alpha}_{\pm}\varphi$ are [34, p. 147]

$$\mathrm{F}(I^{\alpha}_{\pm}\varphi) = (\mp ix)^{-\alpha}\widehat{\varphi}(x) \ \ , \varphi \in L_1(a,b) \ \ . \tag{15}$$

Study of *regular operators* of Mikusiński by Boehme [3] resulted into the theory of Boehmians, the generalization of Schwartz distribution theory. These regular operators form a subalgebra of Mikusiński operators such that they include only such functions whose support is bounded from the left, and at the same time do not have any restriction on the support. The general construction of Boehmians gives rise to various function spaces, which are known as **Boehmian spaces** [cf. Mikusiński and Mikusiński [27] and Mikusiński [28, 29]]. It is observed that these spaces contain all Schwartz distributions, Roumieu ultradistributions and tempered distributions.

The name *Boehmian* is used for all objects by an algebraic construction, which is similar to the construction of the field of quotients. Suppose G is an additive commutative semigroup, S be a subset of group G such that $S \subseteq G$ is a sub semigroup, for which we define a mapping * from $G \times S$ to G such that following conditions are satisfied (these condition are for the mapping *) :

(i) if $\delta, \eta \in S$, then $(\delta * \eta) \in S$ and $\delta * \eta = \eta * \delta$

(ii) if $\alpha \in G, \delta, \eta \in S$, then $(\alpha * \delta) * \eta = \alpha * (\ \delta * \eta)$

(iii) if $\alpha, \beta \in G, \delta \in S$, then $(\alpha + \beta) * \delta = (\alpha * \ \delta) + (\beta * \delta)$.

The delta sequence, denoted by Δ , is defined as members of class delta which are the sequences of subset S, and satisfies the conditions

(i) if $\alpha, \beta \in G, (\delta_n) \in \Delta$ and $(\alpha * \delta_n) = (\beta * \delta_n)$, $\forall\ n$, then $\alpha = \beta$ in G.

(ii) if $(\delta_n), (\varphi_n) \in \Delta$, then $(\delta_n * \varphi_n) \in \Delta$.

Then the *quotient of sequences* is defined as the element of certain class A of pair of sequences defined by

$$A = \{(f_n), (\varphi_n) : (f_n) \subseteq G^N, (\varphi_n) \in \Delta\}.$$

This is denoted f_n/φ_n by such that

$$f_m * \varphi_n = f_n * \varphi_m, \quad \forall\ m, n \in N.$$

Further, the quotients of sequences f_n/φ_n and g_n/ψ_n are called *equivalent* if

$$f_n * \psi_n = g_n * \varphi_n, \qquad \forall\ n \in N.$$

The equivalence relation defined on A and the equivalence classes of quotient of sequence are called *Boehmians* .

The *space of all Boehmians*, denoted by B, has the properties addition, multiplication and differentiation. The Boehmian space B_{L_1} will be called the space of *locally integrable Boehmians* if the group G be the set of all locally integrable function on $\mathbb{R}$ and possibly two such functions are identified with respect to Lebesgue measure (these functions are equal almost everywhere) and the topology of this space is taken to be the semi-norm topology generated by

$$p_n(f) = \int_{-n}^{n} |f|\, d\lambda \quad , n = 1, 2, \dots .$$

where λ is the usual Lebesgue measure on $\mathbb{R}$ and $D(\mathbb{R})$. In other words, if $f \in L_1$ and (δ_n) is the delta sequence, then $\|(f * \delta_n) - f\| \to 0$, as $n \to \infty$. A pair of sequences (f_n, φ_n) is called a quotient of sequences, and is denoted by f_n/φ_n if $f_n \in L_1(n = 1, 2, \dots)$ where (φ_n) is a delta sequence and $f_m * \varphi_n = f_n * \varphi_m, \forall\ m, n \in N$, whereas, two quotients of sequences f_n/φ_n and g_n/ψ_n are equivalent if $f_n * \psi_n = g_n * \varphi_n, \forall\ n \in N$. The equivalence class of quotient of sequences will be called an *integrable Boehmain*, the space of all integrable Boehmian will be denoted by B_{L_1} .Convergence of Boehmians is defined in [Mikusiński [28]]. The terminologies regarding Boehmians and Boehmian spaces can be referred to in [Mikusiński and Mikusiński [27]], Mikusiński [28, 29]. Authors of this paper also investigate the Gabor transform for integrable Boehmian [1], and applications in Fourier and Lapalce transform and distribution spaces to fractional calculus in [19].

8.2 Main Results

Using the relation between the Gabor and the Fourier transform,

$$\mathcal{G}\,|f|\,(\omega, t) = \widetilde{f}_g(\upsilon, t) = \mathrm{F}\{f_t(\tau)\} = \widehat{f}_t(\upsilon)\ ,$$

where F is the Fourier transform and $\mathcal{G}$ is the Gabor transform, respectively. The fractional integrals for the Gabor transform, can be written in the form

$$\mathrm{F}(I^{\alpha}_{\pm} f_t(\tau)) = (\mp i\upsilon)^{-\alpha}\widehat{f}_t(\upsilon) \quad , \ f \in L_1(a, b)\ . \tag{16}$$

In other words, (16) can be written as

$$\mathcal{G}(I^{\alpha}_{\pm} f) = (\mp i\upsilon)^{-\alpha}\widehat{f}_t(\upsilon) \tag{17}$$

i.e.

$$\begin{aligned} \mathcal{G}(I^{\alpha}_{\pm} f_n) &= (\mp i\upsilon)^{-\alpha}(\widehat{f}_t(\upsilon))_n \\ &= (\mp i\upsilon)^{-\alpha}(\widehat{f}_t)_n(\upsilon)\ . \end{aligned} \tag{18}$$

Theorem 1 [20] : If $[f_n/\delta_n] \in B_{L_1}$, then the sequence

$$\mathcal{G}(I^{\alpha}_{\pm} f_n) = (\mp i\upsilon)^{-\alpha}(\widehat{f}_t)_n(\upsilon) \tag{19}$$

converges uniformly on each compact set in $\mathbb{R}$.

Proof : If (δ_n) is a delta sequence, then $(\widehat{\delta}_t)_n$ converges uniformly on each compact set to the constant function unity. Therefore, $(\widehat{\delta}_k) > 0$ on K (the compact set) and, thus, the left hand side of (19) gives

$$\begin{aligned} \mathcal{G}(I^{\alpha}_{\pm} f_n) &= \frac{(I^{\alpha}_{\pm}\widehat{f}_n)(\widehat{\delta}_k)}{(\widehat{\delta}_k)} = \frac{(I^{\alpha}_{\pm} f_n * \delta_k)^{\wedge}}{(\widehat{\delta}_k)} = \frac{(I^{\alpha}_{\pm}\widehat{f}_k)(\widehat{\delta}_n)}{(\widehat{\delta}_k)} \quad \text{on } K \\ &= \frac{(\mp i\upsilon)^{-\alpha}(\widehat{f}_t)_n(\widehat{\delta}_n)}{(\widehat{\delta}_k)} \qquad , \text{[cf. Eqn.(18)]} \end{aligned}$$

This shows that the Gabor transform of fractional integrals for an integrable Boehmian $F = [f_n/\delta_n]$ can be expressed as the limit of the sequence $\mathcal{G}(I^{\alpha}_{\pm} f_n)$, which, in fact, is the space of all continuous functions on $\mathbb{R}$. This proves the theorem completely.

Property 1 [20] : Let $[f_n/\delta_n] \in B_{L_1}$. Then $\Delta - \lim_{n\to\infty} F_n = F$, $\mathcal{G}(I^{\alpha}_{\pm} F_n) \to \mathcal{G}(I^{\alpha}_{\pm} F)$uniformly on each compact set .

Proof : We have $\delta - \lim_{n\to\infty} F_n - F \Rightarrow \mathcal{G}(F_n) \to \mathcal{G}(F)$, uniformly on each compact set. The sequence can be expressed as $F_n * \delta_k, F * \delta_k \in L_1$, $\forall\ n, k \in N$ which has a norm

$$||(F_n - F) * \delta_k|| \to 0, \text{as} \quad n \to \infty, \forall k \in N \quad .$$

where K is well defined. Since $\mathcal{G}\{\delta_k\}$ is a continuous function, we have $\mathcal{G}\{\delta_k\} > 0$ on K for $k \in N$. It is, therefore, enough to prove that

$$\mathcal{G}\{F_n\} \cdot \mathcal{G}\{\delta_k\} \to \mathcal{G}\{F\} \cdot \mathcal{G}\{\delta_k\} \quad ,$$

uniformly on K. We have,

$$\mathcal{G}\{F_n\} \cdot \mathcal{G}\{\delta_k\} - \mathcal{G}\{F\} \cdot \mathcal{G}\{\delta_k\} = \mathcal{G}\{(F_n - F) * \delta_k\} \quad ,$$

such that $||(F_n - F) * \delta_k|| \to 0$, as $n \to \infty$.

This justifies the existence and validity of the property.

Conclusions: The present paper focuses on the application of the Riemann Liouville type fractional integral operator to the Gabor transform and the integrable Boehmians. The fractional integral formula for the Gabor transform is given by using the relation between the Gabor and the Fourier transforms. The formula and the property established in this paper are suitable for certain Boehmian space for an integrable Boehmian. The compact set and the continuity of the function used, approves the existence of the results given in this paper.

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