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FOURIER BESSEL EXPANSION FOR ALEPH-FUNCTION OF SEVERAL VARIABLES II

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Dedicated to Prof. M.A. Pathan on his 75^{th} ***birth anniversary***

Abstract: In this document, we establish one Fourier Bessel expansion for multivariable Aleph-function, I-function of several variables, Aleph-function of two variables and I-function of two variables.

Keywords: Multivariable Aleph-function, Multivariable I-function, Aleph-function of two variables, Fourier Bessel expansion, I-function of two variables.

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1. Introduction and Preliminaries

The object of this paper is to establish one Fourier Bessel expansion for multivariable Aleph-function, I-function of several variables, Aleph-function of two variables and I-function of two variables. The multivariable Aleph-function generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [3]. The generalized multivariable I-function is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.
We have,

$$\aleph(z_1,...,z_r) = \aleph^{0,n:m_1,n_1,...,m_r.n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};...;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}$$

$$\left(\begin{array}{c|l} z_1 & [(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,n}],[\tau_i(a_{ji};\alpha_{ji}^{(1)},...,\alpha_{ji}^{(r)})_{n+1,p_i}]:[(c_j^{(1)},\gamma_j^{(1)})_{1,n_1}], \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & ....[\tau_i(b_{ji};\beta_{ji}^{(1)},...,\beta_{ji}^{(r)})_{m+1,q_i}]:[(d_j^{(1)},\delta_j^{(1)})_{1,m_1}], \end{array}\right.$$

$$\left.\begin{array}{l} [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];...;;[(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \\ [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];...;;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} ... \int_{L_r} \psi(s_1, ..., s_r) \prod_{k=1}^{r} \phi_k(s_k) z_k^{s_k} ds_1...ds_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, ..., s_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} \left[\tau_i \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k) \right]} \tag{1.2}$$

and

$$\phi_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} \left[\tau_{i^{(k)}} \prod_{l=m_k+1}^{q_i(k)} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{l=n_k+1}^{p_i(k)} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k) \right]} \tag{1.3}$$

where $j = 1$ to r and k=1 to r.
Suppose, as usual, that the parameters

$a_j,\ j = 1, ..., p;\ b_j,\ j = 1, ..., q;$

$c_j^{(k)},\ j = 1, ..., n_k; c_{ji^{(k)}}^{(k)},\ j = n_k + 1, ..., p_i(k);$

$d_j^{(k)},\ j = 1, ..., m_k; d_{ji^{(k)}}^{(k)},\ j = m_k + 1, ..., q_i(k);$

with $k = 1$ to r, $i = 1$ to R, $i^{(k)} = 1$ to $R^{(k)}$ are complex numbers, and the $\alpha' s$, $\beta' s$, $\gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_i(k)} \gamma_{ji^{(k)}}^{(k)}$$

$$- \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_i(k)} \delta_{ji^{(k)}}^{(k)} \leq 0 \tag{1.4}$$

The real numbers τ_i are positive for $i = 1$ to R, $\tau_{i(k)}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$.

The contour L_k is in the $s_k - p$ lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop, if necessary, ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as;

$$|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi, \text{ where } A_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} -$$

$$\tau_{i^{(k)}} \sum_{j=n_k+1}^{p_i(k)} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{(q_i(k))} \delta_{ji^{(k)}}^{(k)} > 0, \tag{1.5}$$

with $k = 1, ..., r, i = 1, ..., R$ an $i^{(k)} = 1, ..., R^{(k)}$.
The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R; V = m_1, n_1; ...; m_r, n_r \tag{1.6}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, ..., p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \left\{(a_j; \alpha_j^{(1)}, ..., \alpha_j^{(r)})_{1,n}\right\}, \left\{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, ..., \alpha_{ji}^{(r)})_{n+1,p_i}\right\} \tag{1.8}$$

$$B = \left\{\tau_i(b_{ji}; \beta_{ji}^{(1)}, ..., \beta_{ji}^{(r)})_{m+1,q_i}\right\} \tag{1.9}$$

$$C_1 = \left\{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\right\}, ...,$$

$$C_r = \left\{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\right\} \tag{1.10}$$

$$D_1 = \left\{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\right\}, ...,$$

$$D_r = \left\{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\right\} \tag{1.11}$$

The multivariable Aleph-function write

$$\aleph(z_1,...,z_r) = \aleph^{0,n:V}_{U:W}\left(\begin{array}{c|c} z_1 & A;C_1;...;C_r \\ \cdot & ... \\ \cdot & \\ \cdot & B;D_1;...;D_r \\ z_r & \end{array}\right)$$

2. Multiple Integral

We note

$$W_{31} = p_{i^{(1)}}+3, q_{i^{(1)}}+1, \tau_{i^{(1)}}; R^{(1)}, ..., p_{i^{(r)}}+3, q_{i^{(r)}}+1, \tau_{i^{(r)}}; R^{(r)} \tag{2.1}$$

and

$$V_{11} = m_1+1, n_1+1; ...; m_r+1, n_r+1 \tag{2.2}$$

The multiple integral to be evaluated is

$$\int_0^\infty ... \int_0^\infty x_1^{a_1-1}...x_r^{a_r-1} J_{b_1}(x_1)...J_{b_r}(x_r)\sin x_1 ... \sin x_r \aleph^{0,n:V}_{p_i,q_i,\tau_i;R:W}$$

$$\left(\begin{array}{c|c} z_1x_1^{2c_1} & A;C_1;...;C_r \\ \cdot & ... \\ \cdot & \\ \cdot & B;D_1;...;D_r \\ z_rx_r^{2c_r} & \end{array}\right) dx_1...dx_r = 2^{\sum_{i=1}^r a_i - r} \aleph^{0,n:V_{11}}_{p_i,q_i,\tau_i;R:W_{31}}$$

$$\left(\begin{array}{c|c} 2^{2c_1}z_1 & A:(\frac{2-a_1-b_1}{2};c_1), C_1, (1+b_1-a_1;2c_1), (\frac{1-a_1-b_1}{2};c_1);...; \\ \cdot & ... \qquad ... \\ \cdot & \\ \cdot & B:D_1;(\frac{1}{2}-a_1;c_1)...;...; \\ 2^{2c_r}z_r & \end{array}\right.$$

$$\left.\begin{array}{l} (\frac{2-a_r-b_r}{2};c_r), C_r, (\frac{1-a_r-b_r}{2};c_r), (1+b_r-a_r;2c_r) \\ ... \qquad ... \qquad ... \\ \\ D_r \qquad ..., \qquad (\frac{1}{2}-a_r;c_r) \end{array}\right) \tag{2.3}$$

Provided that

(a) $Re[a_i+b_i+c_i\min\limits_{1\le j\le m_j}\frac{d_j^{(1)}}{\delta_j^{(1)}}] > -1; Re[a_i+2c_j\min\limits_{1\le j\le m_j}\frac{(c_j^{(1)}-1)}{\gamma_j^{(1)}}] < \frac{1}{2}; i=1,...,r.$

(b) $|\arg z_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5).

Proof

To establish (2.1), express the Aleph-function of several variables by the Mellin-Barnes contour type integral (1.1), change the order of integrations (which is permissible under the stated conditions), evaluate the inner-integrals with the help of [1, p.328, (10)]. Finally interpreting the result thus obtained with the Mellin-Barnes contour integral, we arrive at the desired result.

3. Fourier Bessel Expansion

The Fourier series to the established

$$x_1^{a_1}...x_r^{a_r}\sin x_1...\sin x_r \aleph^{0,n:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c} z_1x_1^{2c_1} \\ . \\ . \\ z_rx_r^{2c_r}\end{array}\middle|\begin{array}{c} A;C_1;...;C_r \\ ... \\ B;D_1;...;D_r\end{array}\right)$$

$$=\sum_{m_1=0}^{\infty}...\sum_{m_r=0}^{\infty}\frac{(2m_1+b_1+1)...(2m_r+b_r+1)}{2^{-(\sum_{i=1}^{r}(a_i+2m_i)+r)}}J_{2m_1+b_1+1}(x_1)...J_{2m_r+b_r+1}(x_r)$$

$$\aleph^{0,n:V_{11}}_{p_i,q_i,\tau_i;R:W_{31}}\left(\begin{array}{c} 2^{2c_1}z_1 \\ . \\ . \\ 2^{2c_r}z_r\end{array}\middle|\begin{array}{c} A:(-\frac{b_1+2m_1+a_1}{2};c_1),C_1,(\frac{1-a_1-b_1-2m_1}{2};c_1), \\ ... \quad ... \quad ... \\ B: \quad D_1; \quad ,\end{array}\right.$$

$$\begin{array}{c}(2+b_1+2m_1-a_1;2c_1);...;(-\frac{b_r+2m_r+a_r}{2};c_r),C_r,(\frac{1-a_r-b_r-2m_r}{2};c_r), \\ ... \quad ... \quad ... \quad ... \\ (\frac{1}{2}-a_1;2c_1);...; \quad D_r; \quad ,\end{array}$$

$$\left.\begin{array}{c}(2+b_r+2m_r-a_r;2c_r) \\ ... \\ (\frac{1}{2}-a_r;2c_r)\end{array}\right) \tag{3.1}$$

valid under the conditions of (2.3)

Proof

Let $x_1^{a_1}...x_r^{a^r}\sin x_1...\sin x_r \aleph^{0,n:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c} z_1x_1^{2c_1} \\ . \\ . \\ z_rx_r^{2c_r}\end{array}\right)$

$$=\sum_{m_1=0}^{\infty}...\sum_{m_r=0}^{\infty}A_{m_1,...,m_r}J_{2m_1+b_1+1}(x_1)...J_{2m_r+b_r+1}(x_r) \tag{3.2}$$

Multiplying both sides of (3.2) by $x_1^{-1}...x_r^{-1}J_{2p_1+b_1+1}(x_1)...J_{2p_r+b_r+1}(x_r)$ and integrating with respect to $x_1,...,x_r$ from 0 to ∞ and using (2.3) and orthogonality property of the Bessel functions [2, p. 291, (5) and (6)], we obtain $A_{m_1,...,m_r}$. Substituting the value of $A_{m_1,...,m_r}$, the result (3.1) is obtained.

4. Multivariable I-Function

If $\tau_i = \tau_{i^{(1)}} = ... = \tau_{i^{(r)}} = 1$ the Aleph-function of several variables degenere to the I-function of several variables. The Fourier Bessel expansion have been derived in this section for multivariable I-functions defined by Sharma et al [3]. In these section, we have the following expansion

$$x_1^{a_1}...x_r^{a^r}\sin x_1...\sin x_r I^{0,n:V}_{p_i,q_i;R:W}\left(\begin{array}{c|c} z_1x_1^{2c_1} & A; C_1;...;C_r \\ \cdot & ... \\ \cdot & B; D_1;...;D_r \\ z_rx_r^{2c_r} & \end{array}\right)$$

$$= \sum_{m_1=0}^{\infty}...\sum_{m_r=0}^{\infty}\frac{(2m_1+b_1+1)...(2m_r+b_r+1)}{2^{-(\sum_{i=1}^{r}(a_i+2m_i)+r)}}J_{2m_1+b_1+1}(x_1)...J_{2m_r+b_r+1}(x_r)$$

$$I^{0,n:V_{11}}_{p_i,q_i;R:W_{31}}\left(\begin{array}{c|l} 2^{2c_1}z_1 & A:(-\frac{b_1+2m_1+a_1}{2};c_1), C_1, (\frac{1-a_1-b_1-2m_1}{2};c_1), \\ \cdot & ... \qquad ... \qquad ... \\ \cdot & B: \qquad D_1; \qquad , \\ 2^{2c_r}z_r & \end{array}\right.$$

$$\begin{array}{l}(2+b_1+2m_1-a_1;2c_1);...;(-\frac{b_r+2m_r+a_r}{2};c_r), C_r, (\frac{1-a_r-b_r-2m_r}{2};c_r), \\ ... \qquad ... \qquad ... \qquad ... \\ (\frac{1}{2}-a_1;2c_1);...; \qquad D_r; \qquad , \end{array}$$

$$\left.\begin{array}{l}(2+b_r+2m_r-a_r;2c_r) \\ ... \\ (\frac{1}{2}-a_r;2c_r)\end{array}\right) \tag{4.1}$$

with the same conditions and notations that (2.3)

5. Aleph-Function of Two Variables

If $r = 2$, we obtain the Aleph-function of two variables defined by K. Sharma [5], and we have the following Fourier Bessel expansion.

$$x_1^{a_1}x_2^{a_2}\sin x_1\sin x_2 \aleph^{0,n:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c} z_1x_1^{2c_1} \\ \cdot \\ \cdot \\ z_2x_2^{2c_2}\end{array}\right)$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(2m_1+b_1+1)(2m_2+b_2+1)}{2^{-(\sum_{i=1}^{2}(a_i+2m_i)+2)}}J_{2m_1+b_1+1}(x_1)J_{2m_2+b_2+1}(x_2)$$

$$\aleph_{p_i,q_i,\tau_i;R:W_{31}}^{0,n:V_{11}}\left(\begin{array}{c|l} 2^{2c_1}z_1 & A:(-\frac{b_1+2m_1+a_1}{2};c_1),C_1,(\frac{1-a_1-b_1-2m_1}{2};c_1), \\ \cdot & \cdots \qquad \cdots \qquad \cdots \\ \cdot & B: \qquad D_1; \qquad , \\ 2^{2c_2}z_2 & \end{array}\right.$$

$$\begin{array}{l} (2+b_1+2m_1-a_1;2c_1);(-\frac{b_{+}2m_2+a_2}{2};c_2),C_2,(\frac{1-a_2-b_2-2m_2}{2};c_2), \\ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ (\frac{1}{2}-a_1;2c_1); \qquad D_2; \qquad , \end{array}$$

$$\left.\begin{array}{l} (2+b_2+2m_2-a_2;2c_2) \\ \cdots \\ (\frac{1}{2}-a_2;2c_2) \end{array}\right) \tag{5.1}$$

valid under the same notations and conditions of (2.3) with r=2

6. I-Function of Two Variables

If $\tau_i=\tau_i'=\tau_i''=1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by Sharma et al [4] and we obtain

$$x_1^{a_1}x_2^{a^2}\sin x_1\sin x_2 I_{p_i,q_i;R:W}^{0,n:V}\left(\begin{array}{c} z_1x_1^{2c_1} \\ \cdot \\ \cdot \\ z_2x_2^{2c_2} \end{array}\right)$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(2m_1+b_1+1)...(2m_2+b_2+1)}{2^{-(\sum_{i=1}^{2}(a_i+2m_i)+2)}}J_{2m_1+b_1+1}(x_1)J_{2m_2+b_2+1}(x_2)$$

$$I_{p_i,q_i;R:W_{31}}^{0,n:V_{11}}\left(\begin{array}{c|l} 2^{2c_1}z_1 & A:(-\frac{b_1+2m_1+a_1}{2};c_1),C_1,(\frac{1-a_1-b_1-2m_1}{2};c_1), \\ \cdot & \cdots \qquad \cdots \qquad \cdots \\ \cdot & B: \qquad D_1; \qquad , \\ 2^{2c_2}z_2 & \end{array}\right.$$

$$\begin{array}{l} (2+b_1+2m_1-a_1;2c_1);(-\frac{b_2+2m_2+a_2}{2};c_2),C_2,(\frac{1-a_2-b_2-2m_2}{2};c_2), \\ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ (\frac{1}{2}-a_1;2c_1); \qquad D_2; \qquad , \end{array}$$

$$\left.\begin{array}{l} (2+b_2+2m_2-a_2;2c_2) \\ \cdots \\ (\frac{1}{2}-a_2;2c_2) \end{array}\right) \tag{6.1}$$

valid under the same notations and conditions of (2.3) with r=2

7. Conclusion

The Aleph-function of several variables presented in this paper, is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable I-function, multivariable Fox's H-function, Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function, binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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