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# LAGRANGIANS OF CAWLEY, SUNDERMEYER, AND DI STEFANO

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***Dedicated to Prof. M.A. Pathan on his $75^{th}$ birth anniversary***

**Abstract:** For the Lagrangians of Di Stefano, Sundermeyer, and Cawley we exhibit the Díaz-Higuita-Montesinos expression to calculate the number of physical degrees of freedom.

**Keywords and Phrases:** Lagrangian and Hamiltonian formalisms, Constrained systems.

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## 1. Introduction

In [1] was deduced the following formula to obtain the number of physical degrees of freedom (NPDF) for systems governed by singular Lagrangians:

$$NPDF = N - \frac{1}{2}(l + g + e) \tag{1}$$

where only appear quantities from the Lagrangian formalism, in fact, N, e, l, and g are the total number of generalized coordinates $q_j(t)$, effective gauge parameters [1], genuine constraints and gauge identities [2-5], respectively. This same calculation can be realized with the Hamiltonian expression [6]:

$$NPDF = N - N_1 - \frac{1}{2}N_2 \tag{2}$$

using only concepts from the Rosenfeld-Dirac-Bergmann approach [6-14], where $N_1$ and $N_2$ are the total number of first-and second-class constraints, respectively; let's remember that $N_2$ is an even number [11, 15]. In [1] were established the relations:

$$l = N_1 + N_2 - N_1^{(p)}, \quad g = N_1^{(p)}, \quad e = N_1 \tag{3}$$

being $N_1^{(p)}$ the total number of first-class primary constraints, hence (1) implies (2). On the other hand, it is useful to indicate the connections:

$$M = N - rankW^{(0)}, \quad l = J - M + rankC, \quad N_1^{(p)} = M - rankC \tag{4}$$

where M is the amount of primary constraints (with $M' \leq M$ independent constraints), $W^{(0)}_{NxN}$ is the Hessian matrix, $J = N_1 + N_2$ represents the total number of constraints, and $C_{JxM'} = (\phi_j, \phi_m)$.
In Sec. 2 we apply the matrix and canonical techniques to several Lagrangians studied in [16-18], and thus to exhibit the validity of (1)-(4). To save comments and notations it will be evident when certain quantities are satisfied on shell or on the constraint surface (hence we shall eliminate the usual symbol $\approx 0$).

## 2. Daz-Higuita-Montesinos expression

Here we consider three Lagrangians whose matrix and canonical analysis allows to show the application of the expressions (1)-(4).

$$L = \dot{q}_1\dot{q}_3 + \frac{1}{2}q_3q_2^2, \qquad N = 3. \tag{5}$$

The Lagrangian method [2-5] gives the Hessian matrix

$$W^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

whose rank is 2, with one gauge identity and two genuine constraints:

$$E_2^{(0)} = -q_2q_3, \qquad \varphi^{(0,1)} = q_3, \qquad \varphi^{(1,1)} = \dot{q}_3, \tag{6}$$

hence l = 2, g = 1, and the corresponding local gauge transformation has the following structure:

$$\tilde{q}_1 = q_1, \qquad \tilde{q}_2 = q_2 + \varepsilon\frac{\dot{\beta}}{q_2q_3}, \qquad \tilde{q}_3 = q_3, \qquad \varepsilon \ll 1, \tag{7}$$

where $\beta$ is an arbitrary function, thus $\dot{\beta}$ is an effective gauge parameter, therefore $e = 1$. With the above information the formula (1) given by Daz-Higuita-Montesinos implies that NPDF = 1.

The Hamiltonian approach [8, 9] applied to (5) generates two constraints:

$$\phi_1 = p_2 \quad Primary \quad and \quad \phi_2 = q_2 q_3 \quad Secondary \tag{8}$$

which are of second-class, but it is possible to construct the primary constraint $q_2\phi_1$ of first-class, thus $M = M' = 1$, $N_1 = 1$, $N_2 = 2$, $N_1^{(p)} = 1$, $J = 3$, and $C_{2\times2} \equiv O$ because all Poisson brackets [19-21] $\{\phi_j, \phi_m\}$ worth zero on the constraint region, then rank $C = 0$. With these canonical data the expression (2) implies that $NPDF = 1$, the same value as (1); besides, it is simple to verify the validity of (3) and (4).

$$L = \frac{1}{2} q_1 \dot{q}_2^2 + q_2 q_3, \quad q_1 \neq 0, \quad N = 3. \tag{9}$$

The matrix procedure and (9) lead to

$$W^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

such that rank $W^{(0)} = 1$, with four genuine constraints and one gauge identity:

$$\varphi^{(0,1)} = \dot{q}_2, \quad \varphi^{(0,2)} = q_2, \quad \varphi^{(1,1)} = \dot{q}_1\dot{q}_2 - q_3, \quad \varphi^{(2,1)} = -\dot{q}_3, \quad E_3^{(0)} = -q_2, \tag{10}$$

therefore l = 4, g = 1, and the gauge transformation takes the form:

$$\tilde{q}_1 = q_1, \quad \tilde{q}_2 = q_2, \quad \tilde{q}_3 = q_3 + \varepsilon \frac{\dot{\beta}}{q_2} \tag{11}$$

hence e = 1 because we have one effective gauge parameter; in this case the formula (1) gives NPDF = 0.

The Lagrangian (9) under the canonical method gives two primary constraints:

$$\phi_1 = p_1 \quad First - class, \quad \phi_2 = p_3 \quad Second - class, \tag{12}$$

and three second-class constraints:

$$\phi_3 = p_2 \quad Secondary, \quad \phi_4 = q_2 \quad Secondary, \quad \phi_5 = q_3 \quad Tertiary, \tag{13}$$

then $M = M' = 2$, $N_1 = 1$, $N_2 = 4$, $N_1^{(p)} = 1$, $J = 5$ and rank $C_{5x2} = 1$, thus from (2) is immediate to deduce that $NPDF = 0$ in harmony with (1). The relations (3) and (4) are verified by this set of values generated by the matrix and Hamiltonian approaches.

$$L = \frac{1}{2}(\dot{q}_1^2 + q_1^2 q_2), \quad N = 2. \tag{14}$$

Now rank $W^{(0)} = 1$, with two genuine constraints and one gauge identity:

$$\varphi^{(0,1)} = q_1, \quad \varphi^{(1,1)} = \dot{q}_1, \quad E_2^{(0)} = -\frac{1}{2} q_1^2, \tag{15}$$

that is, l = 2, g = 1, and the gauge transformation is given by:

$$\tilde{q}_1 = q_1, \quad \tilde{q}_2 = q_2 + \varepsilon \frac{\dot{\alpha}}{q_1^2} \tag{16}$$

therefore e = 1; here NPDF = 0 via the Daz-Higuita-Montesinos formula.
For (14) the Hamiltonian formalism leads to one primary, one secondary and one tertiary constraints:

$$\phi_1 = p_2 \quad First-class, \quad \phi_2 = q_1 \quad Second-class, \quad \phi_3 = p_1 \quad Second-class, \tag{17}$$

such that $M = M' = 1$, $N_1 = 1$, $N_2 = 2$, $N_1^{(p)} = 1$, $J = 3$, and rank $C_{3x1} = 0$; thus (2) produces the same value as (1), and (3) and (4) are satisfied.

The aim of this work was to show the use of the relations (1)-(4) with the corresponding set of values from the Lagrangian and Hamiltonian formalisms, and to exhibit the compatibility between the mentioned expressions.

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