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PROPERTY-LOADED VERTEX COLORINGS OF A HYPERGRAPH

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Dedicated to Prof. M.A. Pathan on his 75^{th} ***birth anniversary***

Abstract: Given a hypergraph $H = (X, \mathcal{E})$, an integer $k \geq 1$ and a property $\mathcal{P}$, of subsets of X, a $(\mathcal{P}, k)$-*coloring* of H is a function $\pi : X \rightarrow \{1, 2, \ldots, k\} =: k$ such that for all $i \in k$ the induced subhypergraph $\langle \pi^{-1}(i) \rangle_H \in \bar{\mathcal{P}}$, where $\bar{\mathcal{P}}$ denotes the set of all subsets of X that do not possess the property $\mathcal{P}$. The hypergraph H is $(\mathcal{P}, k)$-*colorable* if and only if it has a $(\mathcal{P}, k)$-coloring. The $\mathcal{P}$-*chromatic number* $\chi_{\mathcal{P}}(H)$ of H is then defined as the least k such that H has a $(\mathcal{P}, k)$-coloring. In this note, we initiate a study of $\chi_{\mathcal{P}}(H)$ for hereditary properties $\mathcal{P}$. For non-hereditary properties, the study appears challenging.

Keywords: hypergraph, coloring, domination, stability, hereditary property, supra-hereditary property, $\mathcal{P}$-*chromatic*, enclaveless set.

AMS Subject Classification: 80 05 C

1. Introduction

For all terminology and notation in the theories of graphs and hypergraphs we refer the reader to Harary [5] and Berge [4], respectively. The hypergraphs considered here are more general in that, unlike in [4], they may have *isolates*, that is, the set Y of vertices that are not contained any edge of the hypergraph; this fundamental difference was first noticed and hypergraphs were treated accordingly in [1].

Hypergraphs are a natural generalization of undirected graphs in which edges may consist of more than 2 vertices. More precisely, a (finite) hypergraph $H = (V, E)$ is a pair $\{X, H\}$ where $H = \{E_1, E_2, \ldots, E_q\}$ is a set of subsets of X such that $E_i \neq \emptyset$ for all i, and $\bigcup_{i=1}^{q} E_i = X$, consisting of p vertices and q edges; if $p = 0$ then H is called the *null hypergraph* and is denoted by K_0. The elements of V are called vertices and the elements of E are called hyper- edges, or simply edges of the hypergraph. A subset S of vertices of H is a *clique* if every two vertices in S are adjacent in H. Thus, it follows that every nonisolated vertex in H is a clique in H; clearly, every subset of a clique is again clique in H. If a hypergraph H has a clique of order $p = |X(H)|$ then it is said to be *complete* and such a graph is denoted by K_p. Clearly, the complete graph $K_p \in \mathcal{K}_p$. Note that K_1, by the definition of a complete hypergraph, is the hypergraph with just one vertex, say x, and just one edge, viz., $\{x\}$; in general, each such edge, with just one vertex, in any hypergraph is called a *self-loop*.

Two vertices u and v are adjacent in $H = (V, E)$ if there is an edge $e \in E$ such that $u, v \in e$. If for two edges $e, f \in E$ holds $e \cap f \neq \emptyset$, we say that e and f are adjacent. A vertex v and an edge e of H are incident if $v \in e$. The degree $deg(v)$ of a vertex $v \in V$ is the number of edges incident to v. The *maximum degree* $max_{v\in V}\ deg(v)$, is denoted by $\triangle(H)$.

The *rank* of a hypergraph $H = (V, E)$ is $r(H) = max_{e\in E}\ |e|$, the anti- rank is $s(H) = min_{e\in E}\ |e|$. A uniform hypergraph H is a hypergraph such that $r(H) = s(H)$. A simple uniform hypergraph of rank r will be called r-uniform. A hypergraph with $r(H) \leq 2$ is a graph. A 2-uniform hypergraph is usually known as a simple graph.

In this paper, we only consider hypergraphs without multiple edges and thus, being E a usual set. If there is a risk of confusion we will denote the vertex set and the edge set of a hypergraph H explicitly by $V(H)$ and $E(H)$, respectively.

A hypergraph $H = (V, E)$ is simple if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E$. The dual H^* of a hypergraph $H = (V, E)$ is the hypergraph whose vertices and edges are interchanged, so that $V(H^*) = \{e_i^* : e_i \in E\}$ and edge set $E(H^*) = \{v_i^* : v_i \in V\}$ with $v_i^* = \{e_j^* : v_i \in e_j\}$

A *partial hypergraph* is obtained by removing a certain number of edges, and removing the nodes that no longer belongs to any hyperedge. Let $J \subset \{1, 2, \ldots, p(H)\}$ and $H = \{E_1, E_2, \ldots, E_q\}$ a hypergraph. Then the partial hypergraph H' is: $H' = \{Ej : E_j \in J\}$. In contrast a *sub-hypergraph* is obtained by removing a subset of the nodes in X, which might result in the removal of edges but in general reduces their size. If $A \subset X$ where X is the node-set belonging to H, $p(H) = p$ then $H_A = \{E_j \bigcap A : 1 \leq j \leq p, E_j \bigcap A \neq \emptyset\}$. Thus the partial hypergraph of H is equal to the sub-hypergraph of the dual hypergraph H^* and vice versa. The partial hypergraph $H' = (V', E')$ is induced if $E' = \{e \in E | e \subseteq V'\}$. Induced hypergraphs will be denoted by $\langle V' \rangle$.

The set $S \subseteq V$ is independent if it contains no edge of E; the maximum cardinality of an independent set is denoted by $\beta(H)$ and is called the *independence number* of H. Some of the older literature, [6, 9] use the term stable and stability number for this concept.

A coloring of a hypergraph H is mapping c from either V or E into a set of colors $C = \{1, \dots, k\}$. We refer to $c : E \to C$ as an edge-coloring and to $c : V \to C$ as a vertex-coloring or simply coloring. A proper coloring of a hypergraph H is a coloring $c : V \to C$ such that $\{v : c(v) = i\}$ is an independent set for all $i \in C$. The *chromatic number* $\chi(H)$ is the minimal number of colors that admit a proper coloring of H. Hence, the chromatic number $\chi(H)$ is the minimum number of independent sets $V_1, \dots, V_{\chi(H)}$ into which V can be partitioned. A *proper strong coloring* of a hypergraph H is a proper coloring such that for all edges $e \in E$ holds that $c(v) \neq c(w)$ for all distinct vertices $v, w \in e$. The *strong chromatic number* $\chi_s(H)$ is the minimal number k of colors that admit a strong k-coloring of H.

A *graph property* $\mathcal{P}$ is *hereditary* if whenever a graph G obeys $\mathcal{P}$, then all induced subgraphs of G obey $\mathcal{P}$ also.

Let $\mathcal{H}$ denote the set of all finite hypergraphs. A subset $\mathcal{P}$ of $\mathcal{H}$ is a *property* if and only if $K_0, K_1 \in \mathcal{P}$; it is *hereditary* if and only if $H \in \mathcal{P}$ and $K \preceq H$ (that is, K is a subhypergraph of H) imply that $K \in \mathcal{P}$; and *nontrivial* if and only if $\mathcal{P} \neq \mathcal{H}$. Clearly, every hereditary hypergraph as defined by Berge [4] is a member of any hereditary property $\mathcal{P}$ and conversely every member of any hereditary property $\mathcal{P}$ is a hereditary hypergraph. If H is any hypergraph with at least two vertices, the set $-H = \{K \in \mathcal{H} : H \not\preceq K\}$ is a hereditary property and any $K \in -H$ is said to be H*-free*. More generally, if $\mathcal{F} \subseteq \mathcal{H}$ and $K_0, K_1 \notin \mathcal{F}$ then the set $-\mathcal{F} = \bigcap_{F \in \mathcal{F}} (-F)$ is a hereditary property.

Next, given a hypergraph $H = (X, \mathcal{E})$ with $\varsigma(H) := \{i \in \mathbf{N} : \exists E \in \mathcal{E}$ such that $|E| = i\}$, $\mathbf{N}$ being the set of natural numbers, we define

$$\bar{H} = (X, \bigcup_{i \in \varsigma(H)} \{\mathcal{X}_i(H) \setminus E : E \in \mathcal{E}\}),$$

where $\mathcal{X}_i(H) = \{A \subseteq X : |A| = i\}$, is called the *complement* of H; for example, $\bar{K_1}$ is the *trivial hypergraph* that consists of just one isolated vertex. Further, if H is a graph then $\varsigma(H) = \{2\}$, whence it is easy to see that $\bar{H}$ is the usual complement of the graph H. Furthermore, if $\mathcal{P}$ is a property then, $\bar{\mathcal{P}} = \{\bar{H} : H \in \mathcal{P}\}$ is a property and if $\mathcal{P}$ is hereditary so is $\bar{\mathcal{P}}$. Also, if $\mathcal{P} = -H$ then $\bar{\mathcal{P}} = -\bar{H}$.

Given a hypergraph $H = (X, \mathcal{E})$, an integer $k \geq 1$ and a property $\mathcal{P}$, of the subsets of X, a $(\mathcal{P}, k)$*-coloring* of H is a function $\pi : X \to \{1, 2, \dots, k\} =: k$ such that for all $i \in k$ the *induced subhypergraph* $\langle \pi^{-1}(i) \rangle_H \in \bar{\mathcal{P}}$; hence any $(\mathcal{P}, k)$-coloring $\pi : X \to k$ of H may be viewed as a partition $P = \{X_1, X_2, \dots, X_k\}$ of $X(H)$ such that for each $i \in k$, X consists of the vertices that are colored by the paint i in which every set X_i is a $\bar{\mathcal{P}}$-set. H is $(\mathcal{P}, k)$*-colorable* if and only if it has a $(\mathcal{P}, k)$-coloring. $\chi_{\mathcal{P}}(H)$ denotes the least positive integer k such that H is $(\mathcal{P}, k)$-colorable. Clearly,

$$\chi_{\mathcal{P}}(H) \leq \chi_s(H) \leq |X(H)| \tag{1}$$

$$\rho_{\mathcal{P}}(H) \leq \omega_{\mathcal{P}}(H) \leq \chi_{\mathcal{P}}(H) \leq \chi(H) \leq |X(H)|, \tag{2}$$

where $\rho_{\mathcal{P}}(H) = max.\{|E| : E \in \mathcal{E} \cap \mathcal{P}\}$ is called the *$\mathcal{P}$-rank* of H and $\omega_{\mathcal{P}}(H)$ denotes the largest order of a clique in H having the property $\mathcal{P}$, or, what we shall call a *$\mathcal{P}$-clique* of H.

Remark 1.1: *Insisting the null hypergraph K_0 to be in $\mathcal{P}$ ensures that any $(\mathcal{P}, k)$-colorable hypergraph is $(\mathcal{P}, m)$-colorable for $m \geq k$.*

Remark 1.2: *Insisting that $K_1 \in \mathcal{P}$ ensures $\chi_{\mathcal{P}}(H) \leq |X(H)|$ for any hypergraph H, since coloring each vertex with a different color is clearly a $\mathcal{P}$-coloring. In particular, $\chi_{\mathcal{P}}(H)$ is always well defined.*

Remark 1.3: *If $\mathcal{P}$ is hereditary, then the restriction of any $(\mathcal{P}, k)$-coloring of a hypergraph H to a subhypergraph K is a $(\mathcal{P}, k)$-coloring of K so that $K \preceq H$ implies $\chi_{\mathcal{P}}(K) \leq \chi_{\mathcal{P}}(H)$.*

Remark 1.4: *For any property $\mathcal{P}$ and hypergraph H, it follows from the definition that*

$$\langle S \rangle_H \in \bar{\mathcal{P}} \Leftrightarrow \langle S \rangle_{\bar{H}} \in \mathcal{P} \tag{3}$$

so that a $(\bar{\mathcal{P}}, k)$-coloring of H is a $(\mathcal{P}, k)$-coloring of $\bar{H}$.

In this note, we initiate a study of $(\mathcal{P}, k)$-colorings of a finite hypergraph and establish a few of their fundamental properties, especially related to the notion of *domination* in hypergraphs (*e.g., see* [2, 3]).

2. An upper bound for $\chi_{\mathcal{P}}(H)$

A natural question to ask is whether there are 'good' bounds for $\chi_{\mathcal{P}}(H)$. The following theorem shows that

$$max.\{\chi(H), \chi(\bar{H})\}$$

is such an upper bound for certain hereditary properties, where $\chi(H)$ denotes the usual *weak chromatic number* of H, defined as the least positive integer k such that one can paint the vertices of H in such a way that every edge of H has at least two vertices of different colors.

Theorem 2.1 *Let $\mathcal{P}$ be a hereditary property. Then $\chi_{\mathcal{P}}(H) \leq max.\{\chi(H),\ \chi(\bar{H})\}$ for all hypergraphs H if and only if $-K_2 \supseteq \mathcal{P}$ or $-\bar{K}_2 \supseteq \mathcal{P}$.*
Proof: If $-K_2 \supseteq \mathcal{P}$ then any usual k-coloring of H is a $(\mathcal{P}, k)$-coloring of H, so that $\chi_{\mathcal{P}}(H) = \chi(H)$. If $-\bar{K}_2 \supseteq \mathcal{P}$ then $-K_2 \supseteq \bar{\mathcal{P}}$, so that $\chi_{\mathcal{P}}(H) = \chi_{\bar{\mathcal{P}}}(\bar{H}) \leq \chi(\bar{H})$. Hence, $\chi_{\mathcal{P}}(H) \leq max.\{\chi(H), \chi(\bar{H})\}$.

Conversely, if $-K_2 \not\supseteq \mathcal{P}$, then as $\mathcal{P}$ is hereditary there must exist complete hypergraphs $S = K_n,\ n \geq 2$ such that $S, \bar{S} \notin \mathcal{P}$. Let $m = n(n-1)$ and M be the disjoint union of m copies $M_1,\ M_2, \ldots, M_m$ of a hypergraph K_m. Then, $\chi(M) = \chi(\bar{M}) = m$, if at least one of the M_i is the complete graph K_m. Now, assume that M is $(\mathcal{P}, m)$-colorable and that $\pi : X(M) \to m$ is such a coloring. Then, as $S \notin \mathcal{P}$ and $\chi(S) = n$ every M_i must be colored with at least n colors so that $\sum_{i=1}^{m} |\pi^{-1}(i)| \geq mn$. However, as $\bar{S} \notin \mathcal{P}$ no color can appear on more than $n-1$ of the M_is, so that $\sum_{i=1}^{m} |\pi^{-1}(i)| \leq m(n-1)$, a contradiction. Thus, $\chi_{\mathcal{P}}(H) > m = max.\{\chi(H), \chi(\bar{H})\}$. Thus, the proof is seen to be complete by contraposition.

It is obvious that the bound is attained by any complete hypergraph K_p and for any hereditary property $\mathcal{P}$. The problem of finding bounds for $\chi_{\mathcal{P}}(H)$ when $\mathcal{P}$ is not hereditary is open.

3. Relation with notions of stability

The set S is stable if it does not contain any edge E with $|E| > 1$ and the stability number $\alpha(H)$ of H is defined as the maximum cardinality of a stable set in H. On the other extreme is the notion of a strongly stable set S in which no two vertices are allowed to be adjacent; in standard graph theory literature, the term independent set is used to mean a s*trongly stable set* and hence we will use the same term for a strongly stable set in a hypergraph. The maximum cardinality of an independent set in H is called its *independence number* and is denoted $\beta(H)$. Clearly, since the set $S_\alpha(H)$ of all maximal stable sets in H contains the set $S_\beta(H)$ of all maximal independent sets in H it follows that $\alpha(H) \geq \beta(H)$ for any hypergraph; also, if H is a graph then equality holds. Given a hypergraph $H = (X, \mathcal{E})$ and a property $\mathcal{P}$ of the subsets of $X(H) := X$, a subset S of $X(H)$ is said to be $\mathcal{P}$-*stable* if

$$E \in \mathcal{E},\quad E \in \mathcal{P} \ \text{ and } \ |E| > 1 \Rightarrow |E \cap S| \leq 1. \tag{4}$$

One may easily note that strongly stable $\Rightarrow$ P-stable $\Rightarrow$ (weakly) stable.

Definition 3.1 *Given any property $\mathcal{P}$ of the subsets of a nonempty set X, a hypergraph $H = (X, \mathcal{E})$ is $\mathcal{P}$-hereditary if for every edge E in H every subset of E also has property $\mathcal{P}$*

Lemma 3.2 *Let $H = (X, \mathcal{E})$ be any $\mathcal{P}$-hereditary hypergraph where $\mathcal{P}$ is a property of the subsets of $X(H)$ such that $\bar{\mathcal{P}}$ is hereditary. Then, in any $(\mathcal{P}, k)$-coloring*

$$P = \{X_1, X_2, \ldots, X_k\}$$

of H, every set X_i is a $\mathcal{P}$-stable set of H.

Proof: Suppose X_i is not a $\mathcal{P}$-stable set of H for some $i \in k$. Then, by definition, it follows that there exists an edge $E \in \mathcal{E}$ such that $E \in \mathcal{P}$, $|E| > 1$, and $|E \cap X_i| \geq 2$. Let $x, y \in E \cap X_i$. since H is $\mathcal{P}$-hereditary and $x, y \in E$ we have $\{x, y\} \in \mathcal{P}$ whereas since $x, y \in X_i$ and X_i is a $\bar{\mathcal{P}}$-set where $\bar{\mathcal{P}}$ is a hereditary property we get $\{x, y\} \in \bar{\mathcal{P}}$ which contradicts the previous deduction. Thus, the result follows.

Lemma 3.3 *Let $H = (X, \mathcal{E})$ be any hypergraph and $\mathcal{P}$ be a hereditary property of the subsets of $X(H)$. Then, in any $(\mathcal{P}, k)$-coloring $P = \{X_1, X_2, \ldots, X_k\}$ of H with $k = \chi_{\mathcal{P}}(H)$, for every two distinct colors $i, j \in \boldsymbol{k}$ there is an edge of H intersecting both X_i and X_j.*

Proof: Suppose the result is false. Then, under the hypotheses, H has a $(\mathcal{P}, k)$-coloring $P = \{X_1, X_2, \ldots, X_k\}$ of H with $k = \chi_{\mathcal{P}}(H)$, in which there are two distinct colors $i, j \in k$ such that no edge of H intersects both X_i and X_j; without loss of generality, we may assume $1 \leq i < j \leq k$. Then, it is not difficult to verify that

$$P' = \{X_1, X_2, \ldots, X_{i-1}, \{X_i \cup X_j\}, X_{i+1}, X_{i+2}, \ldots, X_{j-1}, X_{j+1}, X_{j+2}, \ldots, X_k\}$$

is a $(\mathcal{P}, k-1)$-coloring of H, a contradiction to the minimality of k. Thus, by contraposition, the result is seen to hold.

Next, let $\beta_{\mathcal{P}}(H)$ denote the largest cardinality of a $\mathcal{P}$-stable set in H. Then, for any $(\mathcal{P}, k)$-coloring $P = \{X_1, X_2, \ldots, X_k\}$ of H with $k = \chi_{\mathcal{P}}(H)$, we must have $|X_i| \leq \beta_{\mathcal{P}}(H)$, $\forall i,\ 1 \leq i \leq k$ whence we get $|X(H)| = \sum_{i=1}^{k} |X_i| \leq k \times \beta_{\mathcal{P}}(H)$. Thus, we have

$$\frac{|X(H)|}{\beta_{\mathcal{P}}(H)} \leq \chi_{\mathcal{P}}(H).$$

On the other hand, taking any $\mathcal{P}$-stable set S consisting of $\beta_{\mathcal{P}}(H)$ vertices in H we may construct a $(\mathcal{P}, k)$-coloring by painting all the vertices of S by one color, say c_1, and the remaining $|X(H)| - \beta_{\mathcal{P}}(H)$ vertices by so many distinct colors each different from c_1; we would have thus used $|X(H)| - \beta_{\mathcal{P}}(H) + 1$ distinct colors in all. Existence of this coloring implies

$$\chi_{\mathcal{P}}(H) \leq |X(H)| - \beta_{\mathcal{P}}(H) + 1.$$

Thus, we have the following refinement of the bound in Theorem 2.1 for any $\mathcal{P}$-hereditary hypergraph H, where $\mathcal{P}$ is a property of subsets of $X(H)$ such that $\bar{\mathcal{P}}$ is hereditary.

$$\frac{|X(H)|}{\beta_{\mathcal{P}}(H)} \leq \chi_{\mathcal{P}}(H) \leq |X(H)| - \beta_{\mathcal{P}}(H) + 1. \tag{5}$$

Further, we have the following result as a consequence of Lemma 3.2 and Theorem 2.1

Proposition 3.4 *Let H be a $\mathcal{P}$-hereditary hypergraph where $\mathcal{P}$ is a property of the subsets of $X(H)$ such that $\bar{\mathcal{P}}$ is a hereditary and such that $-K_2 \subseteq \mathcal{P}$ or $-\bar{K}_2 \subseteq \mathcal{P}$. Then*

$$\frac{|X(H)|}{\beta_{\mathcal{P}}(H)} \leq \chi_{\mathcal{P}}(H) \leq min.\{max.\{\chi(H),\ \chi(\bar{H})\}, |X(H)| - \beta_{\mathcal{P}}(H) + 1\}. \tag{6}$$

4. Relation with domination

Let $H = (X, \mathcal{E})$ be any hypergraph, $\mathcal{P}$ be a property of the subsets of $X(H)$ and $D \subseteq X(H)$. We refer to D as a $\mathcal{P}$-*dominating set* of H if D is a $\mathcal{P}$-set and

$$N(x) \cap D \neq \emptyset \ \ \forall x \in X - D, \tag{7}$$

where $N(x) = \{y \in X(H) :\ \exists E \in \mathcal{E} \text{ with } x, y \in E\}$ is called the *vertex neighborhood* of x in H.

Remark 4.1 *P-dominating implies dominating.*

The following known result will be useful to us.

Lemma 4.2 (Acharya, 2001 [2]) *For any hypergraph $H = (X, \mathcal{E})$ and for any property $\mathcal{P}$ of the subsets of $X(H)$, the property $\mathcal{D}$ of the subsets of $X(H)$ being dominating is superhereditary in the sense that every superset of a dominating set in H is a dominating set of H.*

We can now establish the following generalization of a fundamental result of Berge (1973 [4]) in the theory of domination in graphs (*also, see* Walikar *et al.* 1979 [10, 11]; Haynes *et al*, 1997 [7]).

Lemma 4.3 *Let $\mathcal{P}$ be a property of a nonempty set X such that $\bar{\mathcal{P}}$ is hereditary. If $H = (X, \mathcal{E})$ is a $\mathcal{P}$-hereditary hypergraph then every strongly $\mathcal{P}$-stable $\mathcal{P}$-dominating set is a maximal strongly $\mathcal{P}$-stable set. Conversely, for any hypergraph H and for any property $\mathcal{P}$ every maximal strongly $\mathcal{P}$-stable set in H is a minimal $\mathcal{P}$-dominating set.*

Proof: To prove the first part of the theorem, let S be any strongly $\mathcal{P}$-stable $\mathcal{P}$-dominating set in H. Then, there exists a maximal strongly $\mathcal{P}$-stable set T in H such that $S \subseteq T$. If $S = T$ then we are through. Hence, let $S \subset T$. Then, since S is a $\mathcal{P}$-dominating set in H there exists (in fact, for every) vertex $y \in T \setminus S$ such that $N(y) \cap S \neq \emptyset$. Therefore, there exists $x \in S$ such that x and y are adjacent in H. This implies, there exists an edge $E \in \mathcal{E}$ such that $x, y \in E$. Since, by hypothesis, H is a $\mathcal{P}$-hereditary hypergraph, every subset of E must be a $\mathcal{P}$-set and hence the set $\{x, y\}$ is a $\mathcal{P}$-set. However, since $\{x, y\} \subseteq T$ we get a contradiction to our assumption that T is a strongly $\mathcal{P}$-stable set. Thus, S must be a maximal strongly $\mathcal{P}$-stable set of H.

For the converse, let S be a maximal strongly $\mathcal{P}$-stable set in H. If S is not a $\mathcal{P}$-dominating set of H there would exist a vertex $x \in X(H) \setminus S$ such that $N(x) \cap S = \emptyset$ according to the definition. But, then, $S \cup \{x\}$ would be a strongly $\mathcal{P}$-stable set of H, contrary to our assumption that S is a maxiamal strongly $\mathcal{P}$-stable set of H. So, S is a $\mathcal{P}$-dominating set of H.

Next, suppose S is not a minimal $\mathcal{P}$-dominating set of H. This would imply existence of a nonempty proper subset A of S that is a $\mathcal{P}$-dominating set of H. Now, since A is a proper subset of S there exists $y \in S \setminus A$. Since A is $\mathcal{P}$-dominating, by definition, for every $x \in X \setminus A$ there exists $w \in A$ such that x and w are adjacent in H. In particular, since $S \setminus A$ is a subset of $X \setminus A$ we have for $y \in S \setminus A$ there is $a \in A$ such that $a,\ y \in E$ for some $E \in \mathcal{E}$. Also, since $a \in A \subset S$ we get $a,\ y \in E \cap S$, a contradiction to the strong $\mathcal{P}$-stability of S.
This completes the proof.

We are now ready to establish the main result of this section, which generalizes a recent fundamental result obtained by Walikar, *et al.* (2004 [11]).

Theorem 4.4 *Let $H = (X, \mathcal{E})$ be any hypergraph. For a property $\mathcal{P}$ such that $\bar{\mathcal{P}}$ is hereditary, let $P = \{X_1, X_2, \ldots, X_k\}$ be any $(\mathcal{P}, k)$-coloring of the vertices of H, where $k = \chi_{\mathcal{P}}(H)$. Then, either one of the sets in P is a $\mathcal{P}$-dominating set of H or P can be transformed into a $(\mathcal{P}, k)$-coloring $\mathcal{P}'$ of H in which one of the color classes is a $\mathcal{P}$-dominating set of H.*

Proof: Suppose none of the sets in P is a $\mathcal{P}$-dominating set of H. Then, choose any of the sets in P, say X_1. Since it is not $\mathcal{P}$-dominating, there exists $y \in X \setminus X_1$ such that $N(y) \cap X_1 = \emptyset$. Clearly, if $X_1 \cup \{y\}$ is a maximal strongly $\mathcal{P}$-stable set, then it must be a $\mathcal{P}$-dominating set in H by Lemma 4.3. and if $y \in X_j$ then $P_1' = \{X_1 \cup \{y\}, X_2, \ldots, X_j \setminus \{y\}, X_{j+1}, \ldots, X_k\}$ is a $(\mathcal{P}, k)$-coloring of H and we are through because $X_j \setminus \{y\}$ must contain a vertex z adjacent to some vertex in X_1 due to minimality of k, vide Lemma 3.2. If $X_1 \cup \{y\}$ is not a maximal strongly $\mathcal{P}$-stable set of H then there exists $z \in X \setminus (X_1 \cup \{y\})$ such that $X_1 \cup \{y, z\}$ is a strong $\mathcal{P}$-stable set in H. If $z \in X_j \setminus \{y\}$ then there would still exist a vertex w in it such that w is adjacent to some vertex in X_1 since the coloring is complete as $k = \chi_{\mathcal{P}}(H)$ by virtue of Lemma 3.2; clearly, therefore, $w \neq z$.

Thus, $P_2' = \{X_1 \cup \{y, z\}, X_2, \ldots, X_j \setminus \{y, z\}, X_{j+1}, \ldots, X_k\}$ is a $(\mathcal{P}, k)$-coloring of H. On the other hand, if $z \notin X_j \setminus \{y\}$ then $x \in X_r$ for some unique $r \neq j$, $1 \leq r \leq k$ since P is a partition of $X(H)$; without loss of generality, we may assume $r < j$. We may repeat the argument for X_r as in the case of X_j above to conclude that $P_3' = \{X_1 \cup \{y, z\}, X_2, \ldots, X_r \setminus \{z\}, \ldots, X_j \setminus \{y\}, \ldots, X_k\}$ is a $(\mathcal{P}, k)$-coloring of H.

If $X_1 \cup \{y, z\}$ is a maximal strongly $\mathcal{P}$-stable set of H (and hence a $\mathcal{P}$-dominating set) of H, continuing in this manner, since H is finite, it follows that we must eventually obtain a $(\mathcal{P}, k)$-coloring of H in which one of the color classes is a maximal strongly $\mathcal{P}$-stable set set (which must be a $\mathcal{P}$-dominating set) of H. Thus, the proof is seen to be complete.

Let $\gamma_{\mathcal{P}}(H)$ denote the least cardinality of a $\mathcal{P}$-dominating set in H. Theorem 4.4 leads to the following result, which yields a new inequality in the theory of domination in hypergraphs.

Corollary 4.5 *Let $H = (X, \mathcal{E})$ be any hereditary hypergraph where $\mathcal{P}$ is a property of the subsets of the vertex set of $X(H)$ such that $\bar{\mathcal{P}}$ is hereditary. Then*

$$\gamma_{\mathcal{P}}(H) + \chi_{\mathcal{P}}(H) \leq p + 1.$$

In particular, if $\mathcal{P}$ is the property of the subsets of $X(H)$ being *strongly stable* (or, *independent*), that is,

$$S \subset X(H),\ |S| > 1 \Rightarrow |E \cap S| \leq 1 \ \ \forall\ E\ \in \mathcal{E}, \tag{8}$$

or being *weakly stable* in the sense that

$$S \subset X(H),\ |S| > 1 \Rightarrow E \not\subset S \ \ \forall E\ \in \mathcal{E}, \tag{9}$$

(*see* Berge, 1973) then Corollary 4.5 implies

$$\gamma_{si}(H) + \chi_s(H) \leq p + 1 \tag{10}$$

and

$$\gamma_{wi}(H) + \chi_w(H) \leq p + 1, \tag{11}$$

respectively, where $\gamma_{si}(H) := \gamma_i(H)$ is the *independent domination number* of H and $\gamma_{wi}(H)$ is the *weakly stable domination number* of H. The bounds in the above inequalities are all attainable.

5. Relation with enclaveless sets

Given a property $\mathcal{P}$ of the subsets of a nonempty set X and S a set of vertices in a hypergraph $H = (X, \mathcal{E})$, S is $\mathcal{P}$- *enclaveless* (or, $\mathcal{P}$-*full*; cf. Acharya, 2002 [1]) if

$$N_{\mathcal{P}}(x) \cap (X \setminus S) \neq \emptyset, \quad \forall x \in S \tag{12}$$

where

$$N_{\mathcal{P}}(x) = \{y \in X : \ \exists\ E \in \mathcal{E} \ \text{ with } \ E \in \mathcal{P} \ \text{ and } \ x, y \in E\} \tag{13}$$

and x and y are called $\mathcal{P}$-adjacent if there exists a $\mathcal{P}$-edge E of H such that $x, y \in E$. (*see* Slater, 1977 [8]). Let $\mathcal{E}_x$ denote the set of edges in H that contain x called the *edge neighbourhood* of x. A property $\mathcal{P}$ of the subsets of $X(H)$ *acts locally on* H if it contains at least one edge from the edge neighbourhood of each vertex of H.

Remark 5.1 *P-enclaveless implies dominating and P-dominating implies dominating, but P- enclaveless and P-dominating are independent concepts.*

The following result is a variation of a known result (cf. Acharya, 2002 [1]).

Proposition *Let $H = (X, \mathcal{E})$ be a hypergraph and $\mathcal{P}$ be any hereditary property of the subsets of $X(H)$ acting locally on H. Then, every $\mathcal{P}$-stable set in H is $\mathcal{P}$-full.*

Proof: Let S be any $\mathcal{P}$-stable set in H and suppose that it is not $\mathcal{P}$-full. Then, there exists $x \in S$ such that $N_{\mathcal{P}}(x) \cap (X \setminus S) = \emptyset$. Since $\mathcal{P}$ acts locally at every vertex of X there exists a $\mathcal{P}$-edge E containing x such that $E \subset S$. Since $\mathcal{P}$ is hereditary, every subset of E, is also in $\mathcal{P}$. Therefore, if $|E| = 1$ there is nothing to prove. Hence, if $|E| > 1$ then $E \not\subset S$ because S is $\mathcal{P}$-stable. Thus, we get a contradiction.

It is not difficult to construct counterexamples to show that the converse of Proposition 5.2 is not true. The following is a straightforward consequence of Proposition 5.2.

Corollary 5.3 *Let $H = (X, \mathcal{E})$ be a hypergraph and $\mathcal{P}$ be any hereditary property of the subsets of $X(H)$ acting locally on H. Then, $\beta_{\mathcal{P}}(H) \leq f_{\mathcal{P}}(H)$, where $f_{\mathcal{P}}(H)$ denotes the largest cardinality of a $\mathcal{P}$-full set in H.*

The inequality in Corollary 5.3 can be used to refine (3) in certain special cases as the following consequence of Corollary 5.3 shows.

Corollary 5.4 *For any hypergraph $H = (X, \mathcal{E})$ which is locally acted upon by a hereditary property $\mathcal{P}$ of the subsets of $X(H)$ such that $-K_2 \subseteq \mathcal{P}$ or $-\bar{K}_2 \subseteq \mathcal{P}$,*

$$\frac{|X(H)|}{\beta_{\mathcal{P}}(H)} \leq \chi_{\mathcal{P}}(H) \leq min.\{max.\{\chi(H),\ \chi(\bar{H})\}, |X(H)| - f_{\mathcal{P}}(H) + 1\}. \tag{14}$$

Conclusions and scope

As pointed out already, study of the hypergraph properties and parameters discussed in this article need to be pursued in the case of properties which are not hereditary in general. Even in the case of hereditary properties, certain exceptions have been made in the above study and in those exceptional cases too specific studies are necessary to be carried out. Extensions of the property-loaded chromaticity of hypergraphs to the case of weighted hypergraphs are a wider area of investigation and there is a near-future possibility of the results in this direction being useful for real-life applications, especially in mathematical programming and social psychology (*e.g., see* Acharya, 2002, 2003). Lastly, extensions by replacing "hereditary" throughout in the foregoing text by "supra-hereditary" (see [3]) are possible.

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