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# ON DOUBLE SERIES IDENTITIES

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**Abstract:** In this paper, making use of the most generalized form of Bailey's Lemma due to Andrews [2], an attempt has been made to establish certain double series identities.

**Keywords and Phrases:** Double series identities, Bailey's Lemma, multiple series, summation formula.

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## 1. Introduction Notations and Definitions

Throughout this paper we shall adopt the following notation and definition; For any numbers $\alpha$ and q, real or complex and $|q| < 1$, let

$$[\alpha; q]_n \equiv [\alpha]_n = \begin{cases} (1-\alpha)(1-\alpha q)(1-\alpha q^2)...(1-\alpha q^{n-1}) & ; n > 0 \\ 1 & ; n = 0 \end{cases}$$

Accordingly, we have

$$[\alpha; q]_\infty = \prod_{r=0}^{\infty}(1-\alpha q^r).$$

Also,

$$[a_1, a_2, ..., a_r; q]_n \equiv [a_1; q]_n [a_2; q]_n ... [a_r; q]_n.$$

We define a basic hypergeometric series,

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{array}\right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n q^{\lambda n(n-1)/2}}{[q, b_1, b_2, ..., b_s; q]_n}, \quad (|z| < 1). \quad (1.1)$$

Andrews [2] established the following generalized form of Bailey's Lemma.
If,

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q;q]_{n-r}[aq;q]_{n+r}} \tag{1.2}$$

then

$$\sum_{n=0}^{\infty} \frac{[b_1, c_1, b_2, c_2, ..., b_k, c_k, q^{-N}; q]_n}{[aq/b_1, aq/c_1, aq/b_2, aq/c_2, ..., aq/b_k, aq/c_k, aq^{N+1}; q]_n} \times$$
$$\times \left\{ \frac{a^k q^{k+N}}{b_1c_1b_2c_2....b_kc_k} \right\}^n q^{-n(n+1)/2}(-)^n \alpha_n$$
$$= \frac{[aq, aq/b_kc_k; q]_N}{[aq/b_k, aq/c_k; q]_n} \sum_{n_k \geq n_{k-1} \geq n_{k-2} \geq ... \geq n_1 \geq 0} \frac{[b_k, c_k; q]_{n_k} ... [b_1, c_1; q]_{n_1}}{[q;q]_{n_k - n_{k-1}}[q;q]_{n_{k-1}-n_{k-2}} ... [q;q]_{n_2 - n_1}} \times$$
$$\times \frac{[q^{-N}; q]_{n_k}[aq/b_{k-1}c_{k-1}; q]_{n_k - n_{k-1}} ... [aq/b_1c_1; q]_{n_2 - n_1}}{[b_kc_kq^{-N}; q]_{n_k}[aq/b_{k-1}, aq/c_{k-1}; q]_{n_k} ... [aq/b_1, aq/c_1; q]_{n_2}} \times$$
$$\times \frac{q^{n_1+n_2+...+n_k} a^{n_1+n_2+...+n_{k-1}} \beta_n}{(b_{k-1}c_{k-1})^{n_{k-1}} ... (b_1c_1)^{n_1}} \tag{1.3}$$

As $b_1, b_2, ..., b_k, c_k$ and $N \to \infty$ in (1.3), we get

$$\frac{1}{[aq;q]_\infty} \sum_{n=0}^{\infty} q^{kn^2} a^{kn} \alpha_n$$

$$= \sum_{n_k \geq n_{k-1} \geq n_{k-2} \geq ... \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + ... + n_k^2} a^{n_1+n_2+...+n_k} \beta_n}{[q;q]_{n_k - n_{k-1}}[q;q]_{n_{k-1}-n_{k-2}} ... [q;q]_{n_2-n_1}} \tag{1.4}$$

Now, taking $k = 2$ in (1.3) and (1.4), we have
If

$$\beta_n = \frac{1}{[q, aq; q]_n} \sum_{r=0}^{n} \frac{[q^{-n}; q]_r (-q^{n+1})^r \alpha_r}{q^{r(r+1)/2}[aq^{n+1}; q]_r} \tag{1.5}$$

then

$$\sum_{n=0}^{N} \frac{[b_1, c_1, b_2, c_2; q]_n [q^{-N}; q]_n (a^2 q^{N+2}/b_1c_1b_2c_2)^n q^{-n(n-1)/2} (-)^n \alpha_n}{[aq/b_1, aq/c_1, aq/b_2, aq/c_2; q]_n [aq^{N+1}; q]_n}$$
$$= \frac{[aq, aq/b_2c_2; q]_N}{[aq/b_1, aq/c_1; q]_N} \sum_{m \geq n \geq 0} \frac{[b_2, c_2; q]_m [b_1, c_1; q]_n [q^{-N}; q]_m [aq/b_1c_1; q]_{m-n} q^{m+n} a^n \beta_n}{[q;q]_{m-n}[b_2c_2q^{-N}; q]_m [aq/b_1, aq/c_1; q]_m (b_1c_1)^n} \tag{1.6}$$

and

$$\frac{1}{[aq;q]_\infty}\sum_{n=0}^{\infty} q^{2n^2}a^{2n}\alpha_n = \sum_{m,n=0}^{\infty}\frac{a^{m+2n}q^{(m+n)^2+n^2}\beta_n}{[q;q]_m} \tag{1.7}$$

after some simplification.

In this paper we shall utilize (1.5)-(1.7) to establish our main results. Though a number of identities can be established with the help of these result, we shall confine to only a few. We shall use the following summations in our analysis.

$${}_6\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n}; q; aq^{n+1}/bc \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1+n} \end{array}\right] = \frac{[aq, aq/bc; q]_n}{[aq/b, aq/c; q]_n}, \tag{1.8}$$

[Gasper & Rahman 4; App. II (II. 20)]

$${}_4\Phi_3\left[\begin{array}{l} a, -q\sqrt{a}, b, q^{-n}; q; q^{n+1}\sqrt{a}/b \\ -\sqrt{a}, aq/b, aq^{n+1} \end{array}\right] = \frac{[aq, q\sqrt{a}/b; q]_n}{[q\sqrt{a}, aq/b; q]_n}, \tag{1.9}$$

[Gasper & Rahman 4; App. II (II. 14)]

$$\begin{aligned} &{}_4\Phi_3\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, q^{-m}; q; -q^{m-\frac{1}{2}} \\ \sqrt{a}, -\sqrt{a}, aq^{m+1} \end{array}\right] \\ &= \frac{1+\sqrt{a}}{2}\frac{[aq, -q^{-1/2}; q]_m}{[\sqrt{aq}, -\sqrt{a}; q]_m} + \frac{1-\sqrt{a}}{2}\frac{[aq, -q^{-1/2}; q]_m}{[-\sqrt{aq}, \sqrt{a}; q]_m}, \end{aligned} \tag{1.10}$$

[Verma and Jain 6; A (4.1), p. 76]

and

$$\begin{aligned} &{}_2\Phi_1\left[\begin{array}{l} a, q^{-m}; q; -q^{m+\frac{1}{2}} \\ aq^{1+m} \end{array}\right] \\ &= \frac{1+\sqrt{a}}{2}\frac{[aq, -\sqrt{q}; q]_m}{[-\sqrt{aq}, q\sqrt{a}; q]_m} + \frac{1-\sqrt{a}}{2}\frac{[aq, -\sqrt{q}; q]_m}{[\sqrt{aq}, -q\sqrt{a}; q]_m}, \end{aligned} \tag{1.11}$$

[Verma and Jain 6; A (4.3), p. 76]

We shall also use the following set of sequences $<\alpha_n>$ and $<\beta_n>$ in our analysis,
If

$$\alpha_0 = 1, \quad \alpha_n = \frac{(-)^n\left\{(1-aq^{2n})[a;q]_n a^n q^{n(3n-1)/2}\right\}}{\{(1-a)[q;q]_n\}}; \quad n \geq 1$$

then

$$\beta_n = \frac{1}{[q;q]_n} \tag{1.12}$$

and
If

$$\alpha_0 = 1; \quad \alpha_{3n} = \frac{(-)^n [aq^3; q^3]_{n-1}(1-aq^{6n}) a^n q^{3n(3n-1)/2}}{[q^3; q^3]_n},$$

$\alpha_{3n-1} = \alpha_{3n-2} = 0; \quad n \geq 1$ then

$$\beta_0 = 1, \quad \beta_n = \frac{[aq^3; q^3]_{n-1}}{[q;q]_n [aq;q]_{2n-1}}; \qquad n \geq 1 \tag{1.13}$$

[Verma 5; A (4.3), p. 771-72]

## 2. Main Results

In this section we shall establish our main results,

If in (1.5) we set

$$\alpha_r = (-)^r q^{r(r+1)/2} \frac{[a, q\sqrt{a}, -q\sqrt{a}, b, c; q]_r (a/bc)^r}{[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c; q]_r}$$

in (1.5) and apply (1.8), we get,

$$\beta_n = \frac{[aq/bc; q]_n}{[q, aq/b, aq/c; q]_n}$$

Substituting these values of $\alpha_n$ and $\beta_n$ in (1.3), we get

$${}_{10}\Phi_9 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, b_1, c_1, b_2, c_2, q^{-N}; q; a^3 q^{N+3}/bcb_1c_1b_2c_2 \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/b_1, aq/c_1, aq/b_2, aq/c_2, aq^{1+N} \end{matrix} \right]$$

$$= \frac{[aq, aq/b_2c_2; q]_N}{[aq/b_2, aq/c_2; q]_N} \sum_{m=0}^{N} \sum_{n=0}^{m} \frac{[b_2, c_2, q^{-N}]_m q^m [b_1, c_1, aq/bc, q^{-m}; q]_n q^n}{[q, b_2c_2q^{-N}/a; q]_m [q, aq/b, aq/c, b_1c_1q^{-m}/a; q]_n} \tag{2.1}$$

As $N \to \infty$ in the above, we get

$${}_9\Phi_8 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, b_1, c_1, b_2, c_2; q; -a^3q^3/bcb_1c_1b_2c_2 \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/b_1, aq/c_1, aq/b_2, aq/c_2; q \end{matrix} \right]$$

$$= \frac{[aq, aq/b_2c_2; q]_\infty}{[aq/b_2, aq/c_2; q]_\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[b_2, c_2; q]_{m+n} [b_1, c_1, aq/bc; q]_n [aq/b_1c_1; q]_m}{[q;q]_m [q, aq/b, aq/c; q]_n} \times$$

$$\times \frac{(aq/b_2c_2)^m(a^2q^2/b_1c_1b_2c_2)^n}{[aq/b_1c_1;q]_{m+n}} \tag{2.2}$$

Taking $b, c, b_1, c_1, b_2, c_2 \to \infty$ in (2.2) and then letting $a \to 1$, we get

$$\sum_{m,n=0}^{\infty} \frac{q^{2n^2+2mn+m^2}}{[q;q]_m[q;q]_n} = \frac{1}{[q;q]_\infty} \sum_{n=-\infty}^{\infty} (-)^n q^{\frac{7n^2}{2}-\frac{n}{2}} = \frac{1}{[q,q^2,q^5,q^6;q^7]_\infty} \tag{2.3}$$

after some simplification.

This is a known result due to Andrews [1; (1.8), p. 4083].

Next, putting $c_2 = aq^{N+1}$ in (2.1), we get

$${}_8\Phi_7 \left[ \begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, b_1, c_1, b_2; q; a^2q^2/bcb_1c_1b_2 \\ \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/b_1, aq/c_1, aq/b_2 \end{array} \right]_N$$

$$= \frac{[aq, b_2q;q]_N}{[q, aq/b_2;q]_N b_2^N} \sum_{m=0}^{N} \sum_{n=0}^{m} \frac{[b_2, aq^{N+1}, q^{-N};q]_m [b_1, c_1, aq/bc, q^{-m};q]_n q^{m+n}}{[q, b_2q;q]_m [q, aq/b, aq/c, b_1c_1q^{-m}/a;q]_n}. \tag{2.4}$$

Now, setting $b = \sqrt{a}$ and $c = -\sqrt{a}$ in (2.4), we get

$${}_4\Phi_3 \left[ \begin{array}{l} a, b_1, c_1, b_2; q; -aq^2/b_1c_1b_2 \\ \\ aq/b_1, aq/c_1, aq/b_2 \end{array} \right]_N$$

$$= \frac{[aq, b_2q;q]_N}{[q, aq/b_2;q]_N b_2^N} \sum_{m=0}^{N} \sum_{n=0}^{m} \frac{[b_2, aq^{N+1}, q^{-N};q]_m [b_1, c_1, -q, q^{-m};q]_n q^{m+n}}{[q, b_2q;q]_m [q, b_1c_1q^{-m}/a;q]_n [aq^2;q^2]_n}. \tag{2.5}$$

Again, putting $b_1 = q\sqrt{a}, c_1 = -q\sqrt{a}$ and $b_2 = b$ in (2.5), we get

$${}_4\Phi_3 \left[ \begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b; q; 1/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{array} \right]_N$$

$$= \frac{[aq, bq;q]_N}{[q, aq/b;q]_N b^N} \sum_{m=0}^{N} \frac{[b, aq^{1+N}, q^{-N};q]_m q^m}{[q, bq;q]_m} \, {}_2\Phi_1 \left[ \begin{array}{l} -q, q^{-m}; q; q \\ -q^{2-m} \end{array} \right] \tag{2.6}$$

The inner ${}_2\Phi_1$ vanishes for all m except m = 0. Thus we have,

$${}_4\Phi_3 \left[ \begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b; q; 1/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{array} \right]_N = \frac{[aq, bq;q]_N}{[a, aq/b;q]_N b^N}. \tag{2.7}$$

This is yet another known result due to [3; (14), p. 99].
As $N \to \infty$ we get

$$ {}_4\Phi_3 \left[ \begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b; q; 1/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{array} \right] = 0, \tag{2.8}$$

because for convergence; $|b| > 1$.

Again, if we put $b = \sqrt{a}$ or $b = -\sqrt{a}$, respectively in (2.7), we get

$$ {}_2\Phi_1 \left[ \begin{array}{l} a, -q\sqrt{a}; q; 1/\sqrt{a} \\ -\sqrt{a} \end{array} \right]_N = \frac{[aq;q]_N}{[q;q]_N a^{N/2}}. \tag{2.9}$$

and

$$ {}_2\Phi_1 \left[ \begin{array}{l} a, q\sqrt{a}; q; -1/\sqrt{a} \\ \sqrt{a} \end{array} \right]_N = \frac{[aq;q]_N(-)^N}{[q;q]_N a^{N/2}}. \tag{2.10}$$

Now, if we set

$$\alpha_r = q^{r(r+1)/2}(-\sqrt{a}/b)^r \frac{[a, -q\sqrt{a}, b; q]_r}{[q, -\sqrt{a}, aq/b; q]_r}$$

in (1.5) and use (1.9), we get

$$\beta_n = \frac{[q\sqrt{a}/b; q]_n}{[q, q\sqrt{a}, aq/b; q]_n} \tag{2.11}$$

Substituting the above values of $\alpha_n$ and $\beta_n$ in (1.7), we get

$$\frac{1}{[aq;q]_\infty} \sum_{n=0}^{\infty} q^{n(5n+1)/2} a^{2n} (-\sqrt{a}/b)^n \frac{[a, -q\sqrt{a}, b; q]_n}{[q, -\sqrt{a}, aq/b; q]_n}$$
$$= \sum_{m,n=0}^{\infty} \frac{q^{(m+n)^2+n^2} a^{m+2n} [q\sqrt{a}/b; q]_n}{[q;q]_m [q, q\sqrt{a}, aq/b; q]_n} \tag{2.12}$$

As $b \to \infty$ in the above, we get

$$\frac{1}{[aq;q]_\infty} \sum_{n=0}^{\infty} q^{3n^2} a^{5n/2} \frac{[a;q]_n (1 + q^n\sqrt{a})}{[q;q]_n (1 + \sqrt{a})}$$
$$= \sum_{m,n=0}^{\infty} \frac{q^{(m+n)^2+n^2} a^{m+2n}}{[q;q]_m [q;q]_n [q\sqrt{a};q]_n} \tag{2.13}$$

If $a \to 1$ in (2.13), we get

$$\sum_{m,n=0}^{\infty} \frac{q^{2n^2+2mn+m^2}}{[q;q]_m [q;q]_n^2} = \frac{1}{[q;q]_\infty} \tag{2.14}$$

Now, taking $a = q^2$ in (2.13), we get

$$\frac{1}{[q;q]_\infty}\sum_{n=0}^{\infty} q^{3n^2+5n}(1-q^{2n+2}) = \sum_{m,n=0}^{\infty} \frac{q^{(m+n)(m+n+2)+n(n+2)}}{[q;q]_m[q;q]_n[q;q]_{n+1}} \tag{2.15}$$

Again, taking $a = q$ in (2.13), we get

$$\frac{(1-\sqrt{q})}{[q;q]_\infty}\sum_{n=0}^{\infty} q^{3n^2}q^{5n/2}(1+q^{n+\frac{1}{2}}) = \sum_{m,n=0}^{\infty} \frac{q^{(m+n)(m+n+1)+n(n+1)}}{[q;q]_m[q;\sqrt{q}]_{2n}}$$

which leads to

$$\sum_{m,n=0}^{\infty} \frac{q^{(m+n)(m+n+1)+n(n+1)}}{[q;q]_m[q;\sqrt{q}]_{2n}} = \frac{(1-\sqrt{q})}{[q;q]_\infty}[q^6;q^6]_\infty[-q^{11/2},-q^{1/2};q^6]_\infty. \tag{2.16}$$

Further, putting

$$\alpha_r = q^{r(r-1)/2}q^{r/2}\frac{[a;q]_r}{[q;q]_r}$$

in (1.5) and using (1.11), we get

$$\beta_n = \frac{1+\sqrt{a}}{2}\frac{[-\sqrt{q};q]_n}{[q,-\sqrt{aq},q\sqrt{a};q]_n} + \frac{1-\sqrt{a}}{2}\frac{[-\sqrt{q};q]_n}{[q,\sqrt{aq},-q\sqrt{a};q]_n}. \tag{2.17}$$

Substituting the above values of $\alpha_n$ and $\beta_n$ in (1.7), we get

$$\frac{1}{[aq;q]_\infty}\sum_{m,n=0}^{\infty} q^{5n^2/2}a^{2n}\frac{[a;q]_n}{[q;q]_n}$$

$$= \frac{(1+\sqrt{a})}{2}\sum_{m,n=0}^{\infty}\frac{q^{(m+n)^2+n^2}a^{m+2n}[-\sqrt{q};q]_n}{[q;q]_m[q,-\sqrt{aq},q\sqrt{a};q]_n}$$

$$+\frac{(1-\sqrt{a})}{2}\sum_{m,n=0}^{\infty}\frac{q^{(m+n)^2+n^2}a^{m+2n}[-\sqrt{q};q]_n}{[q;q]_m[q,\sqrt{aq},-q\sqrt{a};q]_n} \tag{2.18}$$

which, with $a \to 1$ leads to the identity (2.14).

Next, substituting the values of $\alpha_n$ and $\beta_n$ given by (1.12) and using them in (1.7), we get

$$\frac{1}{[aq;q]_\infty}\left\{1+\sum_{n=1}^{\infty} q^{n(7n-1)/2}a^{2n}(-)^n\frac{(1-aq^{2n})[a;q]_n}{(1-a)[q;q]_n}\right\}$$

$$= \sum_{m,n=0}^{\infty} \frac{q^{(m+n)^2+n^2} a^{m+2n}}{[q;q]_m [q;q]_n} \tag{2.19}$$

As $a \to 1$ in the above, we get

$$\sum_{m,n=0}^{\infty} \frac{q^{(m+n)^2+n^2}}{[q;q]_m [q;q]_n} = \frac{1}{[q,q^2,q^5,q^6;q^7]_\infty} \tag{2.20}$$

which is again the identity (2.3).

Finally, using (1.13) and (1.7), we get

$$\frac{1}{[aq;q]_\infty} \left\{ 1 + \sum_{n=1}^{\infty} q^{3n(15n-1)/2} a^{7n} (-)^n \frac{[a;q^3]_n (1-aq^{6n})}{(1-a)[q^3;q^3]_n} \right\}$$

$$= \sum_{m=0}^{\infty} \frac{a^m q^{m^2}}{[q;q]_m} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{a^{m+2n} q^{(m+n)^2+n^2} [a;q^3]_n}{[q;q]_m [q;q]_n [a;q]_{2n}}$$

As $a \to 1$ in the above we get the following interesting identity, after some simplification

$$= \sum_{m=0}^{\infty} \frac{q^{m^2}}{[q;q]_m} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{q^{(m+n)^2+n^2} [q^3;q^3]_{n-1}}{[q;q]_m [q;q]_n [q;q]_{2n-1}} = \prod_{r \equiv 0, \pm 21 (mod\ 45)} (1-q^r)^{-1}.$$

Making use of several other results available in the literature, a number of interesting identities of the nature investigated here can easily be established.

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