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A NOTE ON CERTAIN IDENTITIES INVOLVING GENERAL DOUBLE AND TRIPLE HYPERGEOMETRIC FUNCTIONS OF EXTON AND SRIVASTAVA

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Abstract: The main object of the present note is to obtain the generalizations and unifications of identities associated with hypergeometric function of one variable ${}_AF_B$ and Appell's hypergeometric function of two variables F_1, F_2, F_3, F_4 due to MacRobert and Sharma respectively, in the form of identities associated with multiple hypergeometric functions $\mathscr{H}$, G of Exton and $F^{(3)}$ of Srivastava.

Keywords and Phrases: Pochhammer's symbol; Multiplication formulae, Double and triple hypergeometric functions; Series iteration technique.

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1. Introduction

In 1967, a unification and generalization of Lauricella's fourteen complete triple hypergeometric functions of second order $F_1, F_2, \cdots, F_{14}$ [8, pp. 113-114] including Saran's ten triple hypergeometric functions F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S,

F_T [12; 13], extended triple hypergeometric function F_K of Sharma[14, p. 613(2)] and three additional triple hypergeometric functions H_A, H_B, H_C of Srivastava [24, pp. 99-100; see also 20; 22; 23; 26], was given by Srivastava [25, p. 428] in the form:

$$F^{(3)}\left[\begin{array}{l} (a_A)\text{::}\ (b_B);(d_D);(e_E)\text{:}\,(g_G);(h_H);(k_K); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x,y,z \\ (\ell_L)\text{::}\,(m_M);(n_N);(p_P)\text{:}\,(r_R);\ (s_S);\ (t_T); \end{array}\right]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{[(a_A)]_{i+j+k}[(b_B)]_{i+j}[(d_D)]_{j+k}[(e_E)]_{k+i}[(g_G)]_i[(h_H)]_j[(k_K)]_k\ x^i\ y^j\ z^k}{[(\ell_L)]_{i+j+k}[(m_M)]_{i+j}[(n_N)]_{j+k}[(p_P)]_{k+i}[(r_R)]_i[(s_S)]_j[(t_T)]_k\ i!\ j!\ k!} \tag{1.1}$$

where the symbol (a_A) abbreviates the array of A parameters $a_1,\ a_2,\ a_3,\cdots,a_A$ in Slater's contracted notation [18, p. 54; 19, p. 41] and Pochhammer's symbol $[(a_A)]_m$ is defined by

$$[(a_A)]_m = \prod_{u=1}^{A}\{(a_u)_m\}$$

$$= \prod_{u=1}^{A}\left\{\frac{\Gamma(a_u+m)}{\Gamma(a_u)}\right\},\quad \text{if}\quad a_u \neq 0,-1,-2,-3,\cdots$$

$$= \prod_{u=1}^{A}\{(a_u)(a_u+1)(a_u+2)\cdots(a_u+m-1)\},\quad \text{if}\quad m=1,2,3,\cdots$$

$$\text{and}\quad (a_u)_0 = 1 \tag{1.2}$$

The notation Γ is used for Gamma function.
The notation $\Delta(n;a)$ denotes the array of n parameters given by

$$\frac{a}{n},\ \frac{a+1}{n},\ \frac{a+2}{n},\cdots,\ \frac{a+n-1}{n};$$

The notation $\Delta[N;(a_A)]$ denotes the array of AN parameters given by

$$\frac{a_1}{N},\ \frac{a_1+1}{N},\ \frac{a_1+2}{N},\cdots,\ \frac{a_1+N-1}{N},\ \frac{a_2}{N},\ \frac{a_2+1}{N},\ \frac{a_2+2}{N},\cdots,\ \frac{a_2+N-1}{N},$$

$$\cdots,\frac{a_A}{N},\ \frac{a_A+1}{N},\ \frac{a_A+2}{N},\cdots,\ \frac{a_A+N-1}{N}$$

with similar interpretation for others.
If $0 \leq j \leq (N-1)$ then the asterisk in $\Delta^*(N;j+1)$ represents the fact the (denominator) parameter $\frac{N}{N}$ is always omitted, so that set $\Delta^*(N;j+1)$ obviously

contains only $(N-1)$ parameters obtained from $\Delta^*(N; j+1)$ because $0 \leq j \leq (N-1)$. Here N is positive integer.
In 1979, Exton [4, p. 339(13)] defined the following double hypergeometric series

$$\mathscr{H}^{A:B;C;D}_{E:G;M;N}\left[\begin{array}{ll} (a_A):(b_B);\ (c_C);(d_D); & \\ & x,\ y \\ (e_E):(g_G);(m_M);(n_N); & \end{array}\right]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{[(a_A)]_{2i+j}\ [(b_B)]_{i+j}\ [(c_C)]_i\ [(d_D)]_j\ x^i\ y^j}{[(e_E)]_{2i+j}\ [(g_G)]_{i+j}\ [(m_M)]_i\ [(n_N)]_j\ i!\ j!} \tag{1.3}$$

It is the generalization of well known Horn's non confluent double hypergeometric functions H_3 [3, p. 225 (15)], H_4 [3, p. 225 (16)], Horn's confluent double hypergeometric functions H_6 [3, p. 226 34)], H_7 [3, p. 226(35)], Appell's four double hypergeometric functions $F_1,\ F_2,\ F_3,\ F_4$ [3, p. 224 (6, 7, 8, 9)], Humbert's seven double hypergeometric $\Phi_1,\ \Phi_2,\ \Phi_3,\ \Psi_1,\ \Psi_2,\ \Xi_1,\ \Xi_2$ [(3, pp. 225-226 (20, 21, 22, 23, 24, 25, 26)], well known Kampé de Fériet's double hypergeometric function [29, p. 423 (26); see also 30, p. 23(1.2, 1.3)]

$$F^{B:C;D}_{G:M;N}\left[\begin{array}{ll} (b_B):\ (c_C);(d_D); & \\ & x,\ y \\ (g_G):(m_M);(n_N); & \end{array}\right]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{[(b_B)]_{i+j}\ [(c_C)]_i\ [(d_D)]_j\ x^i\ y^j}{[(g_G)]_{i+j}\ [(m_M)]_i\ [(n_N)]_j\ i!\ j!}$$

In 1982, Exton [5, p. 137(1.2)] defined additional double hypergeometric function in the following form

$$X^{A:C;D}_{E:M;N}\left[\begin{array}{ll} (a_A):\ (c_C);(d_D); & \\ & x,\ y \\ (e_E):(m_M);(n_N); & \end{array}\right]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{[(a_A)]_{2i+j}\ [(c_C)]_i\ [(d_D)]_j\ x^i\ y^j}{[(e_E)]_{2i+j}\ [(m_M)]_i\ [(n_N)]_j\ i!\ j!} \tag{1.4}$$

In 1984, Exton [6, p. 113(1.2)] defined another double hypergeometric function in the following form

$$G^{A:B;D}_{E:H;M}\left[\begin{array}{ll} (a_A):(b_B);\ \ (d_D); & \\ & x,\ y \\ (e_E):(h_H);(m_M); & \end{array}\right]$$

$$= \sum_{i,j=0}^{\infty} \frac{[(a_A)]_{i-j} \; [(b_B)]_i \; [(d_D)]_j \; x^i \; y^j}{[(e_E)]_{i-j} \; [(h_H)]_i \; [(m_M)]_j \; i! \; j!} \tag{1.5}$$

which is the generalization of Horn's non confluent double hypergeometric functions G_2 [3, p. 224(11)], H_2 [3, p. 225(14)], and Horn's confluent double hypergeometric functions Γ_1, Γ_2, H_2, H_3, H_4, H_5, H_{11} [3, pp. 226-227(27, 28, 30, 31, 32, 33, 39)].
In each hypergeometric form, the denominator parameters are neither zero nor negative integers because Gamma function is not defined for these numbers.
In 1954, MacRobert [9, p. 95(8)] gave the identity in terms of his E-function which is equivalent to the following generalized hypergeometric series identity for ${}_AF_B$

$${}_AF_B\left[\begin{array}{ccc}(a_A) & ; & \\ & & x \\ (e_E) & ; & \end{array}\right] = {}_{2A}F_{2B+1}\left[\begin{array}{ccc}\Delta[2;(a_A)] & ; & \\ & & \frac{x^2}{4^{(1-A+B)}} \\ \frac{1}{2},\ \Delta[2;(b_B)] & ; & \end{array}\right] +$$

$$+\frac{x\prod_{i=1}^{A} a_i}{\prod_{i=1}^{B} b_i}\,{}_{2A}F_{2B+1}\left[\begin{array}{ccc}\Delta[2;(a_A)+1] & ; & \\ & & \frac{x^2}{4^{(1-A-B)}} \\ \frac{3}{2},\ \Delta[2;(b_B)+1] & ; & \end{array}\right] \tag{1.6}$$

The second member of right hand side of identity (1.6) is missing in a paper of Srivastava[21, p. 763(line 8)].
From time to time by the concept of separation of a power series into its even and odd terms, many research workers such as Barr [1, p. 591(1)], Carlson [2, pp. 233-234(10)], Lardner [7, pp. 70-72], Manocha [10, p. 43(3)], Manocha and Jain [11, p. 1479], Sharma [15, pp. 145-146(2)], [16, p. 95(1), p. 99(line 6)], [17, pp. 130-133] and Srivastava [27, p. 191(3)], etc. have also used the hypergeometric series identity (1.6) and its particular cases. In 1974, Sharma [15,pp.148-149,p.153] obtained the hypergeometric series identities corresponding to Appell's double hypergeometric functions F_1, F_2, F_3, F_4.
In 1979, Srivastava [27, p. 194] generalized (1.6) in the following form

$${}_AF_B\left[\begin{array}{ccc}(a_A) & ; & \\ & & x \\ (e_E) & ; & \end{array}\right] = \sum_{j=0}^{N-1} \frac{[(a_A)]_j \; x^j}{[(b_B)]_j \; j!} \times$$

$${}_{NA}F_{NB+N-1}\left[\begin{array}{cc}\Delta(N;a_1+j),\Delta(N;a_2+j),\cdots,\Delta(N;a_A+j); & \\ & \frac{x^N}{N^{(1-A+B)N}} \\ \Delta^*(N;j+1),\Delta(N;b_1+j),\cdots,\Delta(N;b_B+j); & \end{array}\right] \tag{1.7}$$

where N is a positive integer and other notations have their usual meaning.
For $N=2$, (1.7) reduces to (1.6). For $N=3$ it may be remarked that (1.7) reduces to a correct form having the argument $\frac{x^3}{27^{(1-A+B)}}$ in the right hand side in place of incorrect argument $\frac{x^3}{9^{(1-A+B)}}$ of Srivastava [27].
In the derivation of hypergeometric series identities, we shall use the following multiple series identity of Srivastava [27, pp. 196-197; see also 28, p. 217(12)]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}A(m,n,p)=\sum_{j_1=0}^{M-1}\ \sum_{j_2=0}^{N-1}\sum_{j_3=0}^{P-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}A(mM+j_1,nN+j_2,pP+j_3) \tag{1.8}$$

and Pochhammer's symbol identity [28, p. 23(26)]

$$(b)_{MN}=M^{MN}\prod_{u=1}^{M}\left\{\left(\frac{b+u-1}{M}\right)_N\right\},\ \text{where } N=0,1,2,3,\cdots \tag{1.9}$$

We can prove the following Pochhammer's symbol identity by the definition of Pochhammer's symbol (1.2)

$$[(a_A)]_{MN}=M^{MNA}\prod_{u=1}^{M}\left\{\left[\frac{(a_A)+u-1}{M}\right]_N\right\} \tag{1.10}$$

2. Hypergeometric Series Identities

If N, R and C are arbitrary positive integers; numerator, denominator parameters and arguments are adjusted in such a way that each side is completely meaningful, then without any loss of convergence, we have the following three hypergeometric series identities for general double hypergeometric functions G, $\mathscr{H}$ of Exton and general triple hypergeometric function $F^{(3)}$ of Srivastava

$$G^{A:B;D}_{E:H;M}\left[\begin{array}{ll}(a_A):(b_B);(d_D)\ ; & \\ & x,\ y\\ (e_E):(h_H);(m_M)\quad ; & \end{array}\right]$$

$$=\sum_{k=0}^{N-1}\sum_{u=0}^{N-1}\frac{[(a_A)]_{k-u}[(b_B)]_k[(d_D)]_u\ x^k\ y^u}{[(e_E)]_{k-u}[(h_H)]_k[(m_M)]_u\ k!\ u!}\times$$

$$\times G^{NA:NB;ND}_{NE:NH+N-1;NM+N-1}\left[\begin{array}{l}\Delta[N;(a_A)+k-u]:\Delta[N;(b_B)+k]\qquad\qquad ;\\ \\ \Delta[N;(e_E)+k-u]:\Delta[N;(h_H)+k],\Delta^*(N;1+k)\ ;\end{array}\right.$$

$$\left.\begin{matrix} \Delta[N;(d_D)+u]\ ; \\ \Delta[N;(m_M)+u],\ \Delta^*(N;1+u)\ ; \end{matrix} \quad \frac{x^N}{N^{N(1+H+E-A-B)}},\ \frac{y^N}{N^{N(A+M+1-D-E)}} \right] \tag{2.1}$$

$$\mathscr{H}^{A:B;D;E}_{G:M;N;P}\left[\begin{matrix} (a_A):\ (b_B)\ ;(d_D);(e_E)\ ; \\ (g_G):(m_M);(n_N);(p_P)\ ; \end{matrix}\quad x,\ y\right]$$

$$= \sum_{k=0}^{R-1}\sum_{u=0}^{2R-1} \frac{[(a_A)]_{2k+u}[(b_B)]_{k+u}[(d_D)]_k[(e_E)]_u\ x^k\ y^u}{[(g_G)]_{2k+u}[(m_M)]_{k+u}[(n_N)]_k[(p_P)]_u\ k!\ u!}\times$$

$$\times\mathscr{H}^{RB:2RA;2RE;RD}_{RM:2RG;2RP+2R-1;RN+R-1}\left[\begin{matrix} \Delta[R;(b_B)+k+u]\ :\Delta[2R;(a_A)+2k+u]; \\ \Delta[R;(m_M)+k+u]:\Delta[2R;(g_G)+2k+u]; \end{matrix}\right.$$

$$\begin{matrix} \Delta[2R;(e_E)+u]\ ;\ \Delta[R;(d_D)+k]\ ; \\ \Delta[2R;(p_P)+u],\ \Delta^*(2R;1+u)\ ;\ \Delta[R;(n_N)+k],\ \Delta^*(R;1+k); \end{matrix}$$

$$\left.\frac{y^{2R}\ 4^{R(A-G+E-P-1)}}{R^{2R(-A+G-E+P+1-B+M)}},\ \frac{x^R\ 4^{R(A-G)}}{R^{R(-2A+2G-D+N+1-B+M)}}\right] \tag{2.2}$$

$$F^{(3)}\left[\begin{matrix} (a_A)::\ (b_B)\ ;\ (d_D)\ ;\ (e_E)\ :(g_G)\ ;(h_H)\ ;\ (k_K); \\ (\ell_L)\ ::\ (m_M)\ ;\ (n_N)\ ;\ (p_P)\ :\ (r_R)\ ;\ (s_S)\ ;\ \ (t_T); \end{matrix}\quad x,\ y,\ z\right]$$

$$= \sum_{u=0}^{C-1}\sum_{v=0}^{C-1}\sum_{w=0}^{C-1} \frac{[(a_A)]_{u+v+w}[(b_B)]_{u+v}[(d_D)]_{v+w}[(e_E)]_{w+u}[(g_G)]_u[(h_H)]_v[(k_K)]_w x^u y^v z^w}{[(\ell_L)]_{u+v+w}[(m_M)]_{u+v}[(n_N)]_{v+w}[(p_P)]_{w+u}[(r_R)]_u[(s_S)]_v[(t_T)]_w u!v!w!}$$

$$\times F^{(3)}\left[\begin{matrix} \Delta[C;(a_A)+u+v+w]::\Delta[C;(b_B)+u+v]\ ;\Delta[C;(d_D)+v+w]\ ; \\ \Delta[C;(\ell_L)+u+v+w]::\Delta[C;(m_M)+u+v];\ \Delta[C;(n_N)+v+w]; \end{matrix}\right.$$

$$\begin{matrix} \Delta[C;(e_E)+w+u]:\Delta[C;(g_G)+u]\ ; \\ \Delta[C;(p_P)+w+u]:\Delta[C;(r_R)+u],\ \Delta^*(C;1+u)\ ; \end{matrix}$$

$$\Delta[C;(h_H)+u] \qquad ;\Delta[C;(k_K)+w] \qquad ;$$

$$\Delta[C;(s_S)+u],\ \Delta^*(C;1+v)\ ;\Delta[C;(t_T)+w],\ \Delta^*(C;1+w);$$

$$\left.\frac{x^C}{C^{C(L+M+P+R+1-A-B-E-G)}},\frac{y^C}{C^{C(L+M+N+S+1-A-B-D-H)}},\frac{z^C}{C^{C(L+N+P+T+1-A-D-E-K)}}\right] \tag{2.3}$$

which are the generalizations and unifications of hypergeometric series identities (1.6), (1.7) for single hypergeometric function ${}_AF_B$ and hypergeometric series identities of Sharma[15, pp. 148-149, p. 153] for Appell's double hypergeometric functions F_1, F_2, F_3, F_4.

3. Derivations of (2.1), (2.2) and (2.3)

Suppose left hand side of (2.1) is denoted by T, then its power series representation given by

$$T=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{[(a_A)]_{i-j}[(b_B)]_i[(d_D)]_j\ x^i\ y^j}{[(e_E)]_{i-j}[(h_H)]_i[(m_M)]_j\ i!\ j!} \tag{3.1}$$

Now using the following Srivastava's identity [27]

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}F(i,j)=\sum_{k=0}^{N-1}\sum_{u=0}^{N-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}F(iN+k,jN+u), \tag{3.2}$$

after simplification we get

$$T=\sum_{k=0}^{N-1}\sum_{u=0}^{N-1}\frac{[(a_A)]_{k-u}[(b_B)]_k[(d_D)]_u\ x^k\ y^u}{[(e_E)]_{k-u}[(h_H)]_k[(m_M)]_u\ k!\ u!}\times$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{[(a_A)+k-u]_{N(i-j)}[(b_B)+k]_{Ni}[(d_D)+u]_{Nj}\ x^{Ni}\ y^{Nj}}{[(e_E)+k-u]_{N(i-j)}[(h_H)+k]_{Ni}[(m_M)+u]_{Nj}\ (1+k)_{Ni}\ (1+u)_{Nj}} \tag{3.3}$$

Now using (1.9), (1.10) in different Pochhammer's symbols of (3.3), after simplification we have

$$T=\sum_{k=0}^{N-1}\sum_{u=0}^{N-1}\frac{[(a_A)]_{k-u}[(b_B)]_k[(d_D)]_u\ x^k\ y^u}{[(e_E)]_{k-u}[(h_H)]_k[(m_M)]_u\ k!\ u!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\prod_{R=1}^{N}\left\{\left[\frac{(a_A)+k-u+R-1}{N}\right]_{i-j}\right\}}{\prod_{R=1}^{N}\left\{\left[\frac{(e_E)+k-u+R-1}{N}\right]_{i-j}\right\}}\times$$

$$\times \frac{\prod_{R=1}^{N}\left\{\left[\frac{(b_B)+k+R-1}{N}\right]_i\right\}\prod_{R=1}^{N}\left\{\left[\frac{(d_D)+u+R-1}{N}\right]_j\right\}(1)_i(1)_j}{\prod_{R=1}^{N}\left\{\left[\frac{(h_H)+k+R-1}{N}\right]_i\right\}\prod_{R=1}^{N}\left\{\left[\frac{(m_M)+u+R-1}{N}\right]_j\right\}\prod_{R=1}^{N}\left\{\left(\frac{1+k+R-1}{N}\right)_i\right\}}\times$$

$$\times \frac{x^{Ni}\ y^{Nj}}{\prod_{R=1}^{N}\left\{\left(\frac{1+u+R-1}{N}\right)_j\right\}N^{N(1-A-B+E+H)i}\ N^{N(1+A-D-E+M)j}\ i!\ j!} \tag{3.4}$$

Now interpreting the definition of Exton's double hypergeometric function (1.5) in the inner series corresponding to summation indices i, j of (3.4)and having the ideas of the symbols (a_A), $\Delta(N;b)$, $\Delta[N;\ (a_A)]$, $\Delta^*(N;\ j+1)$, $\Delta(N;\ j+1)$, after simplification we have (2.1).

If left hand side of (2.2) is denoted by S, then its power series form is given by

$$S = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{[(a_A)]_{2i+j}[(b_B)]_{i+j}[(d_D)]_i[(e_E)]_j\ x^i\ y^j}{[(g_G)]_{2i+j}[(m_M)]_{i+j}[(n_N)]_i[(p_P)]_j\ i!\ j!} \tag{3.5}$$

Now using the following Srivastava identity[27]

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}F(i,j) = \sum_{k=0}^{R-1}\sum_{u=0}^{2R-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}F(iR+k, 2jR+u), \tag{3.6}$$

after simplification we get

$$S = \sum_{k=0}^{R-1}\sum_{u=0}^{2R-1}\frac{[(a_A)]_{2k+u}[(b_B)]_{k+u}[(d_D)]_k[(e_E)]_u\ x^k\ y^u}{[(g_G)]_{2k+u}[(m_M)]_{k+u}[(n_N)]_k[(p_P)]_u\ k!\ u!}\times$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{[(a_A)+2k+u]_{2R(i+j)}}{[(g_G)+2k+u]_{2R(i+j)}}\times$$

$$\times\frac{[(b_B)+k+u]_{R(2j+i)}[(d_D)+k]_{Ri}[(e_E)+u]_{2Rj}\ x^{Ri}\ y^{2Rj}}{[(m_M)+k+u]_{R(2j+i)}[(n_N)+k]_{Ri}[(p_P)+u]_{2Rj}\ (1+k)_{Ri}\ (1+u)_{2Rj}} \tag{3.7}$$

Now applying the same process in (3.7) as in the derivation of (2.1), we get(2.2).

Similarly if we express the left hand side of (2.3) in power series form, use the following series identity of Srivastava [27]

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{q=0}^{\infty} L(i,j,q) = \sum_{u=0}^{C-1}\sum_{v=0}^{C-1}\sum_{w=0}^{C-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{q=0}^{\infty} L(iC+u, jC+v, qC+w) \quad (3.8)$$

then we get right hand side of (2.3) on the same parallel lines of the derivations of (2.1) and (2.2).

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