

**CERTAIN IDENTITIES RELATED TO THE
ROGERS-RAMANUJAN CONTINUED FRACTION**

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Abstract: In this paper, we employ some theta function identities of Ramanujan to establish certain new theta function identities related to the Rogers-Ramanujan continued fraction. Furthermore, we derive certain general formulas for explicit evaluations of the Rogers-Ramanujan continued fraction.

Keywords and Phrases: Theta-function, modular equations, Rogers-Ramanujan continued fraction.

2000 Mathematics Subject Classification: 11B65, 11A55, 30B70, 11F27, 40A15.

1. Introduction

The well-known Rogers-Ramanujan continued fractions $R(q)$ and $S(q)$ are defined by

$$R(q) := \frac{q^{1/5}}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+\dots} \quad |q| < 1$$

and

$$S(q) := -R(-q).$$

This celebrated continued fraction was first found in the paper by L. J. Rogers in 1894 [15] and he proved that

$$R(q) = q^{1/5} \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty},$$

where

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.$$

S. Ramanujan recorded certain beautiful identities for $R(q)$ and $S(q)$ in his first notebook [13] and in ‘lost’ notebook [14]. For example:

$$\begin{aligned} \frac{1}{R(q)} - 1 - R(q) &= \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}, \\ \frac{1}{S(q)} + 1 - S(q) &= \frac{f(q^{1/5})}{q^{1/5} f(q^5)}, \\ \frac{1}{R^5(q)} - 11 - R^5(q) &= \frac{f^6(-q)}{q f^6(-q^5)}, \\ \frac{1}{S^5(q)} + 11 - S^5(q) &= \frac{f^6(q)}{q f^6(q^5)}, \end{aligned}$$

where $f(-q) = (q; q)_\infty$. For convenience, we set $f(-q^n) = f_n$.

Ramanujan recorded several explicit values for $R(q)$ and $S(q)$ in his notebooks [13], namely

$$\begin{aligned} R(e^{-2\pi}) &= \sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2}, \\ S(e^{-\pi}) &= \sqrt{\frac{5 - \sqrt{5}}{2}} - \frac{\sqrt{5} - 1}{2} \end{aligned}$$

and

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1 + \left[5^{3/4} \left(\frac{\sqrt{5}-1}{2} \right)^{5/2} - 1 \right]^{1/5}} - \frac{\sqrt{5} + 1}{2}.$$

K. G. Ramanathan [12] has evaluated $R(e^{-\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for several rational numbers n using Kronecker’s limit formula. $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for positive rational numbers n has evaluated by B. C. Berndt, H. H. Chan and L. -C. Zhang

[8] in terms of Ramanujan-Weber class invariants and evaluated the same by J. Yi [21] using modular equations relating to degree 5 or 25. Employing Ramanujan's eta-function identities, Berndt and Chan [7] established several values of $R(q)$ and $S(q)$. For some wonderful subsequent works on the Rogers-Ramanujan continued fraction, see [1], [9], [10], [17], [20].

In Section 2, we obtain new modular equations using some modular equations discovered by Ramanujan, K. R. Vasuki and T. G. Ramamurthy [18], Vasuki [16] and Vasuki and G. Sharath [19].

In Section 3, we established certain new general formulas for evaluating $R(q)$ by employing modular equations obtained in Section 2. Further, as a particular case of our general formulas, we deduce certain known numerical values of $R(q)$.

We conclude this introduction by recalling some definitions and identities which we are going to use in coming sections.

The Gauss hypergeometric series ${}_2F_1(a, b; c; x)$ is defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{a(a+1) \cdots (a+n-1) b(b+1) \cdots (b+n-1)}{c(c+1) \cdots (c+n-1) n!} x^n, \quad |x| < 1.$$

Suppose for positive integer n ,

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)} \quad (1.1)$$

holds. The relation between α and β induced by (1.1) is called a modular equation of degree n . We also say β has degree n over α .

Also Ramanujan [13] [4, p. 124], recorded that for

$$q = \exp \left[-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)} \right],$$

$$q^{-1/24} \chi(q) = 2^{\frac{1}{6}} [x(1-x)]^{-1/24}. \quad (1.2)$$

We use the following Schläfli type mixed modular equation recorded by Ramanujan [6, p. 379] and also deduced by N. D. Baruah [2] by employing the identities known to Ramanujan:

Theorem 1.1 Let

$$P_1 := [256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)]^{\frac{1}{48}}$$

and

$$Q_1 := \left[\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)} \right]^{\frac{1}{48}}.$$

Then,

$$\left(Q_1^4 + \frac{1}{Q_1^4} \right) - \left(Q_1^2 + \frac{1}{Q_1^2} \right) - 2 \left(P_1^2 + \frac{1}{P_1^2} \right) = 0, \quad (1.3)$$

where α, β, γ and δ have degrees 1, 5, 7 and 35 respectively.

We also require the following two identities recorded by Ramanujan [13] [4, p. 43]:

If $\alpha\beta = \pi^2$, then

$$e^{\frac{-\alpha}{12}} \sqrt[4]{\alpha} f(-e^{-2\alpha}) = e^{\frac{-\beta}{12}} \sqrt[4]{\beta} f(-e^{-2\beta})$$

and

$$e^{\frac{-\alpha}{24}} \sqrt[4]{\alpha} f(e^{-\alpha}) = e^{\frac{-\beta}{24}} \sqrt[4]{\beta} f(e^{-\beta}).$$

From above two identities, we deduce that if $\alpha\beta = \pi^2$, then

$$e^{\frac{-\alpha}{8}} \sqrt{\alpha} f(e^{-\alpha}) f(-e^{-2\alpha}) = e^{\frac{-\beta}{8}} \sqrt{\beta} f(e^{-\beta}) f(-e^{-2\beta}). \quad (1.4)$$

2. Main results

Theorem 2.1 Let

$$P := \frac{f_1 f_2}{q^{1/2} f_5 f_{10}} \quad \text{and} \quad Q := \frac{f_2 f_4}{q f_{10} f_{20}}.$$

Then,

$$\left(\frac{P}{Q} \right)^4 + \left(\frac{Q}{P} \right)^4 - 3 \left[\left(\frac{P}{Q} \right)^2 + \left(\frac{Q}{P} \right)^2 \right] = \left(PQ + \frac{25}{PQ} \right) \left(\frac{P}{Q} + \frac{Q}{P} \right) + 12.$$

Proof. From [5, p. 206], we have

$$XY + \frac{5}{XY} = \left(\frac{X}{Y} \right)^3 + \left(\frac{Y}{X} \right)^3,$$

where

$$X := \frac{f_1}{q^{\frac{1}{6}} f_5} \quad \text{and} \quad Y := \frac{f_2}{q^{\frac{1}{3}} f_{10}}.$$

Changing q to q^2 in the above, we obtain

$$YY_1 + \frac{5}{YY_1} = \left(\frac{Y}{Y_1}\right)^3 + \left(\frac{Y_1}{Y}\right)^3,$$

where

$$Y_1 := \frac{f_4}{q^{\frac{2}{3}}f_{20}}.$$

Multiplying the above two identities, we obtain

$$\begin{aligned} PQ + \frac{25}{PQ} + 5\left(\frac{P}{Q} + \frac{Q}{P}\right) &= \left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 \\ &+ \left[\frac{XY_1}{Y^2}\right]^3 + \left[\frac{Y^2}{XY_1}\right]^3. \end{aligned} \quad (2.1)$$

From [18], we have

$$X_1^8 B^2 + (X_1 B)^6 - X_1^6 + B^6 + 2(X_1 B)^4 - X_1^2 B^8 + (X_1 B)^2 = 0,$$

where

$$X_1 := q^{\frac{1}{6}} \frac{\chi(q)}{\chi(q^5)} \quad \text{and} \quad B := q^{\frac{1}{3}} \frac{\chi(-q^2)}{\chi(-q^{10})}.$$

Dividing the above by $(X_1 B)^2$, we obtain

$$-\left(\frac{X_1}{B}\right)^3 + \left(\frac{B}{X_1}\right)^3 + \frac{X_1^5}{B} - \frac{B^5}{X_1} + (X_1 B)^3 + 2X_1 B + \frac{1}{X_1 B} = 0.$$

Changing q to $-q$ in the above, we obtain

$$\left(\frac{A}{B}\right)^3 + \left(\frac{B}{A}\right)^3 + \frac{A^5}{B} + \frac{B^5}{A} - (AB)^3 + 2AB - \frac{1}{AB} = 0,$$

where

$$A := q^{\frac{1}{6}} \frac{\chi(-q)}{\chi(-q^5)}.$$

After simplification the above identity reduces to

$$\left(\frac{A}{B}\right)^3 + \left(\frac{B}{A}\right)^3 = \frac{(AB)^2 + \frac{1}{(AB)^2} - 2}{AB + \frac{1}{AB}},$$

which implies

$$\left[\frac{XY_1}{Y^2}\right]^3 + \left[\frac{Y^2}{XY_1}\right]^3 = \frac{(PQ)^2 + \frac{1}{(PQ)^2} - 2}{PQ + \frac{1}{PQ}}.$$

Using this in (2.1), we obtain the required result.

Theorem 2.2. [13, p. 213], [5, p. 230]. Let

$$P := \frac{f_1 f_2}{q^{\frac{1}{2}} f_5 f_{10}} \quad \text{and} \quad Q := \frac{f_3 f_6}{q^{\frac{3}{2}} f_{15} f_{30}}.$$

Then,

$$PQ + \frac{25}{PQ} = \left[\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2\right] - 3\left[\frac{Q}{P} + \frac{P}{Q}\right] - 6. \quad (2.2)$$

Recently K. R. Rajanna [11, p. 49] deduced (2.2) using theta function identities.

Lemma 2.1. Let

$$P := q^{\frac{3}{2}} \frac{f_1 f_{50}}{f_5 f_{10}} \quad \text{and} \quad Q := q^{\frac{1}{2}} \frac{f_2 f_{25}}{f_{10} f_5}.$$

Then,

$$\frac{1}{PQ} - 1 = \left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q} + \frac{Q}{P}\right).$$

Proof. From [13, p. 325] [5, p. 206], we have

$$A_1 A_2 + \frac{5}{A_1 A_5} = \left(\frac{A_2}{A_1}\right)^3 + \left(\frac{A_1}{A_2}\right)^3, \quad (2.3)$$

where

$$A_n := \frac{f_n}{q^{\frac{n}{6}} f_{5n}}.$$

Changing q to q^5 in (2.3), we obtain

$$A_5 A_{10} + \frac{5}{A_5 A_{10}} = \left(\frac{A_{10}}{A_5}\right)^3 + \left(\frac{A_5}{A_{10}}\right)^3.$$

Multiplying (2.3) and the above identity, we obtain

$$A_1 A_2 A_5 A_{10} + \frac{25}{A_1 A_2 A_5 A_{10}} + 5 \left[PQ + \frac{1}{PQ} \right] =$$

$$\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 + \left[\frac{A_2 A_5}{A_1 A_{10}}\right]^3 + \left[\frac{A_1 A_{10}}{A_2 A_5}\right]^3. \quad (2.4)$$

From [14, p. 55], we have

$$(B_1 B_5)^2 + \frac{4}{(B_1 B_5)^2} = \left(\frac{B_5}{B_1}\right)^3 - \left(\frac{B_1}{B_5}\right)^3, \quad (2.5)$$

where

$$B_n := \frac{f_n}{q^{\frac{n}{24}} f_{2n}}.$$

For a proof of (2.5) one may refer [3], [11].

Changing q to q^5 in (2.5), we obtain

$$(B_5 B_{25})^2 + \frac{4}{(B_5 B_{25})^2} = \left(\frac{B_{25}}{B_5}\right)^3 - \left(\frac{B_5}{B_{25}}\right)^3.$$

Multiplying (2.5) and the above identity, we obtain

$$\begin{aligned} (B_1 B_5^2 B_{25})^2 + \frac{16}{(B_1 B_5^2 B_{25})^2} + 4 \left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 \right] = \\ \left[\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 \right] - \left[\left(\frac{A_1 A_{10}}{A_2 A_5}\right)^3 + \left(\frac{A_2 A_5}{A_1 A_{10}}\right)^3 \right]. \end{aligned} \quad (2.6)$$

From [5, p. 207], we have

$$A_1 A_2 - \frac{5}{A_1 A_5} = (B_1 B_5)^2 - \frac{4}{(B_1 B_5)^2}. \quad (2.7)$$

Changing q to q^5 in the above, we have

$$A_5 A_{10} - \frac{5}{A_5 A_{10}} = (B_5 B_{25})^2 - \frac{4}{(B_5 B_{25})^2}.$$

Multiplying the above two identities, we obtain

$$A_1 A_2 A_5 A_{10} + \frac{25}{A_1 A_2 A_5 A_{10}} - 5 \left[PQ + \frac{1}{PQ} \right] =$$

$$(B_1 B_5^2 B_{25})^2 + \frac{16}{(B_1 B_5^2 B_{25})^2} - 4 \left[\left(\frac{P}{Q} \right)^2 + \left(\frac{Q}{P} \right)^2 \right]. \quad (2.8)$$

Subtracting (2.8) from (2.4), we obtain

$$\begin{aligned} 10 \left[PQ + \frac{1}{PQ} \right] + (B_1 B_5^2 B_{25})^2 + \frac{16}{(B_1 B_5^2 B_{25})^2} - 4 \left[\left(\frac{P}{Q} \right)^2 + \left(\frac{Q}{P} \right)^2 \right] = \\ \left[\left(\frac{P}{Q} \right)^3 + \left(\frac{Q}{P} \right)^3 \right] + \left[\left(\frac{A_1 A_{10}}{A_2 A_5} \right)^3 + \left(\frac{A_2 A_5}{A_1 A_{10}} \right)^3 \right]. \end{aligned} \quad (2.9)$$

Now adding (2.6) and (2.9), we obtain

$$(B_1 B_5^2 B_{25})^2 + \frac{16}{(B_1 B_5^2 B_{25})^2} + 5 \left[PQ + \frac{1}{PQ} \right] = \left[\left(\frac{P}{Q} \right)^3 + \left(\frac{Q}{P} \right)^3 \right]. \quad (2.10)$$

We have from [16]

$$\left[(B_1 B_5^2 B_{25})^2 + \frac{16}{(B_1 B_5^2 B_{25})^2} \right] = \frac{C(P, Q)}{D(P, Q)},$$

where

$$\begin{aligned} C(P, Q) = \left(\frac{P}{Q} \right)^5 + \left(\frac{Q}{P} \right)^5 - 6 \left[\left(\frac{P}{Q} \right)^4 + \left(\frac{Q}{P} \right)^4 \right] + 11 \left[\left(\frac{P}{Q} \right)^3 + \left(\frac{Q}{P} \right)^3 \right] \\ - 16 \left[\left(\frac{P}{Q} \right)^2 + \left(\frac{Q}{P} \right)^2 \right] + 21 \left[\frac{P}{Q} + \frac{Q}{P} \right] - 30 \end{aligned}$$

and

$$D(P, Q) = 1 - \left[\frac{P}{Q} + \frac{Q}{P} \right] + \left[\left(\frac{P}{Q} \right)^2 + \left(\frac{Q}{P} \right)^2 \right].$$

Using the above in (2.10) and then factoring, we obtain

$$M(P, Q)N(P, Q) = 0,$$

where

$$M(P, Q) = P^4 + Q^4 - P^3Q - PQ^3 + (PQ)^2 - PQ$$

and

$$N(P, Q) = P^4 + Q^4 - P^3Q - Q^3P + (PQ)^2 - (PQ)^3.$$

By definition of P and Q , we see that

$$P = q^{3/2} [1 - q - q^2 + 2q^5 - q^6 + 4q^{10} \dots]$$

and

$$Q = q^{1/2} [1 - q^2 - q^4 + q^5 - q^7 - q^9 + 4q^{10} \dots]$$

Using these in $M(P, Q)$ and $N(P, Q)$, we find that

$$M(P, Q) = -3q^{13} + 3q^{14} - 9q^{16} + 8q^{17} + \dots$$

and

$$N(P, Q) = q^2 - q^3 - 2q^4 + q^5 - q^6 + 7q^7 + \dots$$

Now $q^{-2}N(P, Q) \neq 0$ as $q \rightarrow 0$, where as $q^{-2}M(P, Q) \rightarrow 0$ as $q \rightarrow 0$ thus $q^{-2}M(P, Q) = 0$ in the neighborhood of $q = 0$. Hence by analytic continuation,

$$M(P, Q) = P^4 + Q^4 - P^3Q - PQ^3 + (PQ)^2 - PQ = 0$$

for all values of q with $|q| < 1$.

Divide the above identity by P^2Q^2 to obtain the required result.

Theorem 2.3. Let

$$P := \frac{f_1 f_2}{q^{\frac{1}{2}} f_5 f_{10}} \quad \text{and} \quad Q := \frac{f_5 f_{10}}{q^{\frac{5}{2}} f_{25} f_{50}}.$$

Then,

$$\begin{aligned} (PQ)^2 + \left(\frac{25}{PQ}\right)^2 - 15 \left[PQ + \frac{25}{PQ}\right] + 5PQ \left[\frac{Q}{P} + 25\frac{P}{Q}\right] + 143 = \\ \left(\frac{Q}{P}\right)^3 - 21 \left(\frac{Q}{P}\right)^2 + 97 \left(\frac{Q}{P}\right). \end{aligned} \quad (2.11)$$

Proof. From [14, p. 325] [5, p. 213], we have

$$PQ + \frac{25}{PQ} = \left(\frac{K}{L}\right)^3 + \left(\frac{L}{K}\right)^3 - 4 \left[\left(\frac{K}{L}\right)^2 + \left(\frac{L}{K}\right)^2\right], \quad (2.12)$$

where

$$K := \frac{f_1}{qf_{25}} \quad \text{and} \quad L := \frac{f_2}{q^2f_{50}}.$$

From Lemma 2.1, we have

$$\frac{Q}{P} - 1 = \left(\frac{K}{L}\right)^2 + \left(\frac{L}{K}\right)^2 - \left(\frac{L}{K} + \frac{K}{L}\right).$$

Put $\frac{K}{L} + \frac{L}{K} = x$ in the above identity, we obtain

$$\frac{Q}{P} + 1 = x^2 - x.$$

Substitute the above in (2.12) to obtain the required result.

Lemma 2.2. Let

$$K := \frac{qf_1f_{35}}{f_5f_7} \quad \text{and} \quad L := \frac{q^2f_2f_{70}}{f_{10}f_{14}}.$$

Then,

$$KL + \frac{1}{KL} = \left(\frac{L}{K}\right)^3 + \left(\frac{K}{L}\right)^3 + 4 \left[\left(\frac{L}{K}\right)^2 + \left(\frac{K}{L}\right)^2 \right] + 8 \left[\frac{L}{K} + \frac{K}{L} \right] + 12.$$

Proof. See [19].

Lemma 2.3. Let

$$M := \frac{f_1f_{14}}{q^{\frac{5}{2}}f_5f_{70}} \quad \text{and} \quad N := \frac{f_2f_7}{q^{\frac{3}{2}}f_{10}f_{35}}.$$

Then

$$\begin{aligned} & \left[\left(\frac{M}{N}\right)^2 + \left(\frac{N}{M}\right)^2 \right] + \left[\left(\frac{M}{N}\right)^3 + \left(\frac{N}{M}\right)^3 \right] + \left[\left(\frac{M}{N}\right)^4 + \left(\frac{N}{M}\right)^4 \right] \\ & + 2 \left(\frac{M}{N} + \frac{N}{M} \right) \left[\left(\frac{M}{N}\right)^2 + \left(\frac{N}{M}\right)^2 \right] = \left(MN + \frac{25}{MN} \right) + 4. \end{aligned}$$

Proof. Changing q to q^7 in (2.3), we obtain

$$A_7A_{14} + \frac{5}{A_7A_{14}} = \left(\frac{A_{14}}{A_7}\right)^3 + \left(\frac{A_7}{A_{14}}\right)^3.$$

Multiplying (2.3) and the above identity and then using the fact that $M = A_1A_{14}$ and $N = A_2A_7$, we obtain

$$\begin{aligned} & \left(MN + \frac{25}{MN} \right) + 5 \left[\frac{A_1A_2}{A_7A_{14}} + \frac{A_7A_{14}}{A_1A_2} \right] \\ &= \left(\frac{M}{N} \right)^3 + \left(\frac{N}{M} \right)^3 + \left(\frac{A_2A_{14}}{A_1A_7} \right)^3 + \left(\frac{A_1A_7}{A_2A_{14}} \right)^3. \end{aligned} \quad (2.13)$$

Changing q to q^7 in (2.5), we see that

$$(B_7B_{35})^2 + \frac{4}{(B_7B_{35})^2} = \left(\frac{B_{35}}{B_7} \right)^3 - \left(\frac{B_7}{B_{35}} \right)^3.$$

Multiplying (2.5) and the above identity and then using the fact that $\frac{M}{N} = \frac{B_1B_{35}}{B_5B_7}$ and $\frac{A_1A_7}{A_2A_{14}} = \frac{B_1B_7}{B_5B_{35}}$, we find that

$$\begin{aligned} & \left[(B_1B_5B_7B_{35})^2 + \frac{16}{(B_1B_5B_7B_{35})^2} \right] + 4 \left[\left(\frac{B_1B_5}{B_7B_{35}} \right)^2 + \left(\frac{B_7B_{35}}{B_1B_5} \right)^2 \right] \\ &= \left[\left(\frac{A_2A_{14}}{A_1A_7} \right)^3 + \left(\frac{A_1A_7}{A_2A_{14}} \right)^3 \right] - \left[\left(\frac{M}{N} \right)^3 + \left(\frac{N}{M} \right)^3 \right]. \end{aligned} \quad (2.14)$$

Changing q to q^7 in (2.7), we see that

$$A_7A_{14} - \frac{5}{A_7A_{14}} = (B_7B_{35})^2 - \frac{4}{(B_7B_{35})^2}.$$

Multiplying (2.7) and the above identity, we obtain

$$\begin{aligned} & \left(MN + \frac{25}{MN} \right) - 5 \left[\frac{A_1A_2}{A_7A_{14}} + \frac{A_7A_{14}}{A_1A_2} \right] \\ &= (B_1B_5B_7B_{35})^2 + \frac{16}{(B_1B_5B_7B_{35})^2} - 4 \left[\left(\frac{B_1B_5}{B_7B_{35}} \right)^2 + \left(\frac{B_7B_{35}}{B_1B_5} \right)^2 \right]. \end{aligned} \quad (2.15)$$

Adding (2.13) and (2.15), we obtain

$$2 \left(MN + \frac{25}{MN} \right) - \left[(B_1B_5B_7B_{35})^2 + \frac{16}{(B_1B_5B_7B_{35})^2} \right]$$

$$\begin{aligned}
& +4 \left[\left(\frac{B_1 B_5}{B_7 B_{35}} \right)^2 + \left(\frac{B_7 B_{35}}{B_1 B_5} \right)^2 \right] \\
& = \left[\left(\frac{A_2 A_{14}}{A_1 A_7} \right)^3 + \left(\frac{A_1 A_7}{A_2 A_{14}} \right)^3 \right] + \left[\left(\frac{M}{N} \right)^3 + \left(\frac{N}{M} \right)^3 \right]. \quad (2.16)
\end{aligned}$$

Subtracting (2.14) from (2.16), we find that

$$\left(MN + \frac{25}{MN} \right) - 4 \left[\left(\frac{B_1 B_5 B_7 B_{35}}{2} \right)^2 + \left(\frac{2}{B_1 B_5 B_7 B_{35}} \right)^2 \right] = \left[\left(\frac{M}{N} \right)^3 + \left(\frac{N}{M} \right)^3 \right]. \quad (2.17)$$

Also employing (1.2) in (1.3), we obtain

$$\begin{aligned}
& 4 \left[\left(\frac{B_1 B_5 B_7 B_{35}}{2} \right)^2 + \left(\frac{2}{B_1 B_5 B_7 B_{35}} \right)^2 \right] = \left[\left(\frac{M}{N} \right)^4 + \left(\frac{N}{M} \right)^4 \right] \\
& + \left[\left(\frac{M}{N} \right)^2 + \left(\frac{N}{M} \right)^2 \right] + 2 \left[\left(\frac{M}{N} \right)^2 + \left(\frac{N}{M} \right)^2 \right] \left[\frac{M}{N} + \frac{N}{M} \right] - 4. \quad (2.18)
\end{aligned}$$

Substitute (2.18) in (2.17) to obtain the required result.

Theorem 2.4. Let

$$P := \frac{f_1 f_2}{q^{\frac{1}{2}} f_5 f_{10}} \quad \text{and} \quad Q := \frac{f_7 f_{14}}{q^{\frac{7}{2}} f_{35} f_{70}}.$$

Then,

$$\begin{aligned}
& \left[(PQ)^3 + \frac{25^3}{(PQ)^3} \right] + 7 \left[PQ + \frac{25}{PQ} \right] + 714 \\
& + \left[7 \left(\frac{25^2}{(PQ)^2} + (PQ)^2 \right) + 28 \left(\frac{25}{PQ} + PQ \right) - 98 \right] \left[\frac{P}{Q} + \frac{Q}{P} \right] \\
& + \left[28 \left(\frac{25}{PQ} + PQ \right) + 84 \right] \left[\left(\frac{P}{Q} \right)^2 + \left(\frac{Q}{P} \right)^2 \right] \\
& + 28 \left[\left(\frac{P}{Q} \right)^3 + \left(\frac{Q}{P} \right)^3 \right] - \left[\left(\frac{P}{Q} \right)^4 + \left(\frac{Q}{P} \right)^4 \right] = 0. \quad (2.19)
\end{aligned}$$

Proof. Using $PQ=MN$, $\frac{P}{Q}=KL$ and $\frac{M}{N} = \frac{K}{L} = x$ (say) in Lemma 2.2 and Lemma 2.3, then eliminate x between obtained two equations to deduce the required result.

3. Evaluations

Lemma 3.1. For $q = e^{-\pi\sqrt{n/5}}$, let

$$J_n = \frac{f(q)f(-q^2)}{\sqrt{5} q^{\frac{1}{2}} f(q^5)f(-q^{10})}.$$

Then

$$J_n J_{\frac{1}{n}} = 1, \quad (3.1)$$

$$J_1 = 1, \quad (3.2)$$

$$\frac{\varphi^2(q)}{\varphi^2(q^5)} = \frac{J_n^2 - 5 + \sqrt{J_n^4 - 6J_n^2 + 25}}{2}, \quad (3.3)$$

$$\text{where } \varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}$$

$$\frac{x^2(5-x)}{x-1} = \frac{1}{S^5(q)} + 11 - S^5(q) \quad (3.4)$$

and

$$\frac{x(5-x)^2}{(x-1)^2} = \frac{1}{R^5(q^2)} - 11 - R^5(q^2). \quad (3.5)$$

Proof. Using the definition of J_n and $J_{\frac{1}{n}}$ in the left side of (3.1) and applying (1.4) twice, we readily obtain (3.1). Setting $n = 1$ in (3.1), we obtain (3.2).

From [1], we have

$$\frac{H^2(q)[5-H(q)]}{H(q)-1} \times \frac{H(q)[5-H(q)]^2}{[H(q)-1]^2} = \left(\frac{f_1 f_2}{q^{\frac{1}{2}} f_5 f_{10}} \right)^6,$$

where

$$H(q) = \frac{\varphi^2(-q)}{\varphi^2(-q^5)}.$$

Changing q to $-q$ in the above identity and then setting $q = e^{-\pi\sqrt{n/5}}$, we deduce that

$$x^2 - (J_n^2 - 5)x - J_n^2 = 0, \quad \text{where } x = H(-q).$$

Solving the above equation, we obtain (3.3). For a proof of (3.4) and (3.5) see [1].

Theorem 3.1. We have

$$5 \left[J_n J_{9n} + \frac{1}{J_n J_{9n}} \right] = \left(\frac{J_{9n}}{J_n} \right)^2 + \left(\frac{J_n}{J_{9n}} \right)^2 + 3 \left[\frac{J_{9n}}{J_n} + \frac{J_n}{J_{9n}} \right] - 6. \quad (3.6)$$

Proof. Changing q to $-q$ in (2.2), we obtain

$$RS + \frac{25}{RS} = \left(\frac{S}{R} \right)^2 + \left(\frac{R}{S} \right)^2 + 3 \left[\frac{S}{R} + \frac{R}{S} \right] - 6,$$

where

$$R := \frac{f(q)f(-q^2)}{q^{\frac{1}{2}}f(q^5)f(-q^{10})} \quad \text{and} \quad S := \frac{f(q^3)f(-q^6)}{q^{\frac{3}{2}}f(q^{15})f(-q^{30})}.$$

It is easy to see that $R = \sqrt{5}J_n$ and $S = \sqrt{5}J_{9n}$ for $q = e^{-\pi\sqrt{n/5}}$. Using these in the above, we obtain the required result.

Theorem 3.2. We have

$$J_3 = \sqrt{\frac{3 + \sqrt{5}}{2}} \quad \text{and} \quad J_{\frac{1}{3}} = \sqrt{\frac{3 - \sqrt{5}}{2}}.$$

Proof. Setting $n = \frac{1}{3}$ in (3.6) and employing (3.1), we obtain

$$J_3^4 - 3J_3^2 + 1 = 0.$$

Solving the above equation and observing $J_n > 0$, we obtain

$$J_3 = \sqrt{\frac{3 + \sqrt{5}}{2}}.$$

Since $J_n J_{\frac{1}{n}} = 1$, we obtain

$$J_{\frac{1}{3}} = \sqrt{\frac{3 - \sqrt{5}}{2}}.$$

Theorem 3.3. We have

$$J_9 = 2 + \sqrt{3} \quad \text{and} \quad J_{\frac{1}{9}} = 2 - \sqrt{3}$$

Proof. Setting $n = 1$ in (3.6) and employing (3.2), we obtain

$$J_9^4 - 4J_9 + 1 = 0.$$

Solving the above equation and observing J_n is an increasing function, we obtain

$$J_9 = 2 + \sqrt{3}.$$

Since $J_n J_{\frac{1}{n}} = 1$, we obtain

$$J_{\frac{1}{9}} = 2 - \sqrt{3}.$$

Theorem 3.4. We have

$$\begin{aligned} & 25 \left[(J_n J_{25n})^2 + \frac{1}{(J_n J_{25n})^2} \right] + 75 \left[J_n J_{25n} + \frac{1}{J_n J_{25n}} \right] \\ & - 25 J_n J_{25n} \left[\frac{J_{25n}}{J_n} + 25 \frac{J_n}{J_{25n}} \right] + 143 = \left(\frac{J_{25n}}{J_n} \right)^3 - 21 \left(\frac{J_{25n}}{J_n} \right)^2 + 97 \left(\frac{J_{25n}}{J_n} \right). \end{aligned}$$

Proof. Changing q to $-q$ in (2.11), we obtain

$$\begin{aligned} (RS)^2 + \frac{(25)^2}{RS} + 15 \left[RS + \frac{1}{RS} \right] - 5RS \left[\frac{S}{R} + 25 \frac{R}{S} \right] + 143 = \\ \left(\frac{S}{R} \right)^3 - 21 \left(\frac{S}{R} \right)^2 + 97 \left(\frac{S}{R} \right), \end{aligned}$$

where

$$R := \frac{f(q)f(-q^2)}{q^{\frac{1}{2}}f(q^5)f(-q^{10})} \quad \text{and} \quad S := \frac{f(q^5)f(-q^{10})}{q^{\frac{5}{2}}f(q^{25})f(-q^{50})}.$$

It is easy to see that $R = \sqrt{5}J_n$ and $S = \sqrt{5}J_{25n}$ for $q = e^{-\pi\sqrt{n/5}}$. Using these in the above, we obtain required result.

Theorem 3.5. We have

$$\begin{aligned} & 125 \left[(J_n J_{49n})^3 + \frac{1}{(J_n J_{49n})^3} \right] + 35 \left[J_n J_{49n} + \frac{1}{J_n J_{49n}} \right] + 714 \\ & - \left[175 \left(\frac{1}{(J_n J_{49n})^2} + (J_n J_{49n})^2 \right) + 140 \left(\frac{1}{J_n J_{49n}} + J_n J_{49n} \right) - 98 \right] \\ & \times \left[\frac{J_{49n}}{J_n} + \frac{J_n}{J_{49n}} \right] + \left[140 \left(\frac{1}{J_n J_{49n}} + J_n J_{49n} \right) + 84 \right] \left[\left(\frac{J_{49n}}{J_n} \right)^4 + \left(\frac{J_n}{J_{49n}} \right)^4 \right] \\ & - 28 \left[\left(\frac{J_{49n}}{J_n} \right)^3 + \left(\frac{J_n}{J_{49n}} \right)^3 \right] - \left[\left(\frac{J_{49n}}{J_n} \right)^4 + \left(\frac{J_n}{J_{49n}} \right)^4 \right] = 0. \end{aligned} \quad (3.7)$$

Proof. Changing q to $-q$ in (2.19), we obtain

$$\begin{aligned}
 & \left[(RS)^3 + \frac{25^3}{(RS)^3} \right] + 7 \left[RS + \frac{25}{RS} \right] + 714 \\
 & + \left[7 \left(\frac{25^2}{(RS)^2} + (RS)^2 \right) + 28 \left(\frac{25}{RS} + RS \right) - 98 \right] \left[\frac{R}{S} + \frac{S}{R} \right] \\
 & + \left[28 \left(\frac{25}{RS} + RS \right) + 84 \right] \left[\left(\frac{R}{S} \right)^2 + \left(\frac{S}{R} \right)^2 \right] \\
 & + 28 \left[\left(\frac{R}{S} \right)^3 + \left(\frac{S}{R} \right)^3 \right] - \left[\left(\frac{R}{S} \right)^4 + \left(\frac{S}{R} \right)^4 \right] = 0.
 \end{aligned} \tag{3.8}$$

where

$$R := \frac{f(q)f(-q^2)}{q^{\frac{1}{2}}f(q^5)f(-q^{10})} \quad \text{and} \quad S := \frac{f(q^7)f(-q^{14})}{q^{\frac{7}{2}}f(q^{35})f(-q^{70})}.$$

It is easy to see that $R = \sqrt{5}J_n$ and $S = \sqrt{5}J_{49n}$ for $q = e^{-\pi\sqrt{n/5}}$. Using these in the above, we obtain required result.

Theorem 3.6. We have

$$J_7 = \sqrt{\frac{A^2 + 16 + 5A + \sqrt{A^4 + 48A^2 + 966 + 30\sqrt{105} + 160A}}{3A}}$$

and

$$J_{\frac{1}{7}} = \sqrt{\frac{A^2 + 16 + 5A - \sqrt{A^4 + 48A^2 + 966 + 30\sqrt{105} + 160A}}{3A}},$$

where

$$A = \left(71 + 3\sqrt{105} \right)^{\frac{1}{3}}.$$

Proof. Setting $n = \frac{1}{7}$ in (3.7), we obtain

$$t^4 + 28t^3 - 368t^2 + 448t - 304 = 0,$$

where

$$t = J_7^2 + \frac{1}{J_7^2}.$$

Solving the above equation and substituting $t = J_7^2 + \frac{1}{J_7^2}$, we obtain

$$J_7^4 - \left[\frac{2A}{3} + \frac{32}{3A} + \frac{10}{3} \right] J_7^2 + 1 = 0.$$

Solving this, we obtain the required result J_7 . Since $J_n J_{\frac{1}{n}} = 1$, we also obtain $J_{\frac{1}{7}}$.

Theorem 3.7. We have

$$J_{49} = \frac{1}{3B} \left[B^2 + 32B + 889 + \sqrt{B^4 + 2793B^2 + 56896B + 2502129 + 59136\sqrt{15}} \right]$$

and

$$J_{\frac{1}{49}} = \frac{1}{3B} \left[B^2 + 32B + 889 - \sqrt{B^4 + 2793B^2 + 56896B + 2502129 + 59136\sqrt{15}} \right],$$

where

$$B = \left(26747 + 924\sqrt{15} \right)^{\frac{1}{3}}.$$

Proof. Setting $n = 1$ in (3.7), we obtain

$$t^4 - 62t^3 + 52t^2 + 88t - 544 = 0,$$

where

$$t = J_{49} + \frac{1}{J_{49}}.$$

Solving the above equation and substituting $t = J_{49} + \frac{1}{J_{49}}$, we obtain

$$J_{49}^2 - \left(\frac{2B}{3} + \frac{1778}{3B} + \frac{64}{3} \right) J_{49} + 1 = 0.$$

Solving this, we obtain the required result J_{49} . Since $J_n J_{\frac{1}{n}} = 1$, we also obtain $J_{\frac{1}{49}}$.

References

- [1] C. Adiga, K. R. Vasuki and K. Shivasankara, Some theta function identities and new explicit evaluations of Rogers-Ramanujan continued fraction, Tamsui Oxford Jour. Math. Sci., **18**(1) (2002), 101-117.
- [2] N. D. Baruah, On some of Ramanujan's Schläfli-type mixed modular equations, J. Number theory, **100** (2003), 270-294.

- [3] B. C. Berndt, Modular equations in Ramanujan's 'Lost' Notebook, in Number Theory, R. P. Bambah, V. C. Dumir and R. J. Hans-Gill eds., Hindustan Book Agency, New Delhi, 2000, 55-74.
- [4] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer - Verlag, New York, 1991.
- [5] B. C. Berndt, Ramanujan's Notebooks, Part IV, Springer - Verlag, New York, 1994.
- [6] B. C. Berndt, Ramanujan's Notebooks, Part V, Springer - Verlag, New York, 1998.
- [7] B. C. Berndt and H. H. Chan, Some values for the Rogers - Ramanujan continued fraction, Canadian J. Math., **47** (1995), 897-914.
- [8] B. C. Berndt, H. H. Chan and L. -C. Zhang, Explicit evaluations of the Rogers - Ramanujan continued fraction, J. Reine Angew. Math., **480** (1996), 141-159.
- [9] H. H. Chan and V. Tan, On the explicit evaluations of the Rogers - Ramanujan continued fraction, Contemporary Mathematics, **236** (1999), 127-136.
- [10] S. Y. Kang, Ramanujan's formulas for the explicit evaluation of the Rogers - Ramanujan continued fraction and theta-functions, Acta Arithmetica, **90**(1) (1999), 49-68.
- [11] K. R. Rajanna, An Elucidative Study in the field of q -series and continued fractions, Doctorial Thesis, University of Mysore, 2009.
- [12] K. G. Ramanathan, Some applications of Kronecker's limit formula, J. Indian Math. Soc., **52** (1987), 71-89.
- [13] S. Ramanujan, Notebooks, Vol. II, Tata Institute of Fundamental Research, Bombay, 1957.
- [14] S. Ramanujan, The Lost Notebook and other unpublished papers, Narosa, New Delhi, 1988.
- [15] L. J. Rogers, On a type of modular relation, Proc. London Math. Soc., **19** (1921), 387-397.

- [16] K. R. Vasuki, On certain Ramanujan - Weber type modular equations, J. Indian Math. Soc., **75**(1-4) (2008), 173-192.
- [17] K. R. Vasuki and M. S. Mahadeva Naika, Some evaluations of the Rogers - Ramanujan continued fractions, Proceedings of the International Conference on The Works of Srinivasa Ramanujan, July 1-2 (2000), 209-217.
- [18] K. R. Vasuki and T. G. Sreeramamurthy, Certain new Ramanujan's Schläfli-type mixed modular equations, J. math. Anal. Appl., **309** (2005), 238-255.
- [19] K. R. Vasuki and G. Sharath, On Ramanujan's modular equations of degree 35, J. Math. Anal. Appl., **396** (2012), 13-20.
- [20] K. R. Vasuki and K. Shivashankara, Some new values for the Rogers - Ramanujan continued fraction, J. Indian Math. Soc., **69** (2002), 1-11.
- [21] J. Yi, Evaluations of the Rogers - Ramanujan continued fraction $R(q)$ by modular equations, Acta Arithmetica, **97**(2) (2001), 103-127.

