

ON FOURIER-TRANSFORM, GENERALIZED FOURIER TRANSFORM AND THEIR APPLICATIONS

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Abstract: In this paper Fourier transform, generalized Fourier transform and their applications have been discussed. q-generalizations of these transforms have also been established.

Keywords and Phrases: Fourier transform, generalized Fourier transform, q-integral, q-exponential function.

2000 AMS subject classification: 44A10, 47D03.

1. Introduction

We define the Fourier transform of the function $f(x)$ as,

$$F(k) = \mathcal{F}[f(x); k] = \int_{-\infty}^{\infty} f(x)e^{-ikx} dx, \quad (1.1)$$

provided, of course, that this integral exists. Sometimes it is also represented by,

$$F(k) = \mathcal{F}[f(x); k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx. \quad (1.2)$$

The Fourier transform was introduced by Fourier in 1811 but the notion of it as a transform did not occur in his work. Cauchy in 1816 produced a more mathematical and complete description of this Fourier transform. Here we shall discuss some properties of Fourier transform.

(a) We shall now consider the Fourier transform of $f(ax)$, where $a > 0$ is a real constant.

$$\mathcal{F}[f(ax); k] = \int_{-\infty}^{\infty} f(ax) e^{-ikx} dx \quad (1.3)$$

Putting $ax = z, dx = \frac{dz}{a}$ in (1.3) we have

$$\begin{aligned} \mathcal{F}[f(ax); k] &= \frac{1}{a} \int_{-\infty}^{\infty} f(z) e^{-ikz/a} dz \\ &= \frac{1}{a} \mathcal{F}[f(x); k/a]. \end{aligned} \quad (1.4)$$

This is called change of scale property.

(b) If we take the function $f(x - a)$ then

$$\mathcal{F}[f(x - a); k] = \int_{-\infty}^{\infty} f(x - a) e^{-ikx} dx. \quad (1.5)$$

Taking $x - a = z, dx = dz$ in (1.5) we have,

$$\begin{aligned} \mathcal{F}[f(x - a); k] &= \int_{-\infty}^{\infty} f(z) e^{-ik(z+a)} dz \\ &= e^{-ika} \int_{-\infty}^{\infty} f(z) e^{-ikz} dz \\ &= e^{-ika} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= e^{-ika} \mathcal{F}[f(x); k]. \end{aligned} \quad (1.6)$$

This is called the shifting property of Fourier transform.

(c) Taking the function $e^{-ik'x} f(x)$ we have,

$$\begin{aligned} \mathcal{F}[e^{-ik'x} f(x); k] &= \int_{-\infty}^{\infty} e^{-ik'x} f(x) e^{-ikx} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-i(k+k')x} dx \\ &= \mathcal{F}[f(x); k + k']. \end{aligned} \quad (1.7)$$

(d) We shall consider the Fourier transform of the derivative of $f(x)$.

$$\mathcal{F}[f'(x); k] = \int_{-\infty}^{\infty} f'(x)e^{-ikx} dx = [e^{-ikx} f(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} ike^{-ikx} f(x) dx \quad (1.8)$$

For the existence of the Fourier transform we require $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Hence,

$$\mathcal{F}[f'(x); k] = ik \int_{-\infty}^{\infty} f(x)e^{-ikx} dx = ik\mathcal{F}[f(x); k]. \quad (1.9)$$

We can generalize (1.9) in the following form,

$$\mathcal{F}[f^n(x); k] = (ik)^n \mathcal{F}[f(x); k]. \quad (1.10)$$

(e) The inverse Fourier transform of (1.1) is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)e^{ikx} dk. \quad (1.11)$$

2. q-Generalization of the Fourier and Inverse Fourier Transform

For $|q| < 1$, the q-shifted factorial is defined as,

$$(a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}), \quad \text{for } n = 1, 2, 3, \dots, \infty.$$

Also,

$$(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_r; q)_n,$$

$$(a_1, a_2, \dots, a_r; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \dots (a_r; q)_\infty$$

and

$$(a; q)_\infty = \prod_{r=0}^{\infty} (1 - aq^r). \quad (2.1)$$

With these notations, we define the basic hypergeometric series as,

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n z^n}{(q, b_1, b_2, \dots, b_s; q)_n} \{(-1)^n q^{n(n-1)/2}\}^{1+s-r}, \quad (2.2)$$

which is convergent for $|z| < \infty$ if $1 + s > r$ and $|z| < 1$ if $1 + s = r$.

There are two q-exponential functions,

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1. \quad (2.3)$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} z^n}{(q; q)_n} = (-z; q)_{\infty}. \quad (2.4)$$

For a fixed q , the q -derivative D_q is defined by,

$$D_q\{g(z)\} = \frac{g(z) - g(zq)}{(1-q)z}$$

and

$$D_q(z^n) = \frac{z^n - (zq)^n}{z(1-q)} = \frac{1 - q^n}{1-q} z^{n-1}. \quad (2.5)$$

The basic analogue of Fourier transform or q -Fourier transform is defined as,

$$F_q(k) = F_q[f(x); k] = \int_{-\infty}^{\infty} f(x) e_q^{ikx(f(x))^{q-1}} d_q x, \quad 1 \leq q \leq 3, \quad (2.6)$$

where e_q^z denotes the principal values of $[1 + (1-q)z]^{1/(1-q)}$ and as $q \rightarrow 1$, $e_q^z \rightarrow e^z$.

The Inverse Fourier Transform is defined as;

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} F_q\{f(x); k\} d_q k, \quad (2.7)$$

where the q -integral is defined by,

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x, \quad (2.8)$$

$$\int_0^a f(x) d_q x = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \quad (2.9)$$

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n. \quad (2.10)$$

and

$$\int_{-\infty}^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} \{f(q^n) + f(-q^n)\} q^n. \quad (2.11)$$

3. Two Variables Generalization of Fourier Transform

Taking $x = (x_1, x_2)$, $t = (t_1, t_2)$, $xt = x_1 t_1 + x_2 t_2$, $dt = dt_1 dt_2$ the two variables generalization of Fourier Transform is defined by,

$$F(x) = \mathcal{F}[f(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixt} f(t) dt,$$

which can be also expressed as,

$$F(x_1, x_2) = \mathcal{F}[f(t_1, t_2)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix_1 t_1 - ix_2 t_2} f(t_1, t_2) dt_1 dt_2. \quad (3.1)$$

The function $F(x_1, x_2)$ is called Fourier Transform of the function $f(t_1, t_2)$. Corresponding Inverse Fourier Transform is defined as,

$$f(t) = \mathcal{F}^{-1}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixt} F(x) dx$$

which can be written as,

$$f(t_1, t_2) = \mathcal{F}^{-1}[f(x_1, x_2)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix_1 t_1 + ix_2 t_2} F(x_1, x_2) dx_1 dx_2. \quad (3.2)$$

n-variables generalization of Fourier Transform can be similarly given as,

If $x, t \in R^n$ i.e. $x = (x_1, x_2, \dots, x_n)$, $t = (t_1, t_2, \dots, t_n)$

$$xt = x_1 t_1 + x_2 t_2 + \dots + x_n t_n \quad \text{and} \quad dt = dt_1 dt_2 \dots dt_n$$

then

$$F(x) = \mathcal{F}\{f(t)\} = \frac{1}{(2\pi)^{1/n}} \int_{R^n} e^{-ixt} f(t) dt$$

which takes the following form if written in full-form,

$$\begin{aligned} F(x_1, x_2, \dots, x_n) &= \mathcal{F}\{f(t_1, t_2, \dots, t_n)\} \\ &= \frac{1}{(2\pi)^{1/n}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-ix_1 t_1 - ix_2 t_2 - \dots - ix_n t_n} f(t_1, t_2, \dots, t_n) dt_1 dt_2 \dots dt_n. \end{aligned} \quad (3.3)$$

Corresponding inverse transform is defined as,

$$\begin{aligned} f(t) &= \mathcal{F}^{-1}\{F(x_1, x_2, \dots, x_n)\} \\ &= \frac{1}{(2\pi)^{1/n}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{ix_1 t_1 + ix_2 t_2 + \dots + ix_n t_n} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n. \end{aligned} \quad (3.4)$$

For detail about Fourier Transform and their generalizations one is referred the books [1],[2] and [3] given in the references.

References

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