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ON CERTAIN CLASS OF EULER TYPE INTEGRALS INVOLVING EXTENDED AND MULTIPARAMETER HURWITZ LERCH ZETA FUNCTIONS

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Abstract: In this paper we establish some new class of Beta integrals for functions involving extended and multi-parameter Hurwitz-Lerch Zeta functions and hypergeometric functions. Our results would generalize and extend the work by Srivastava[10] and Bin-Saad[1]. We also obtain certain known and unknown new results as applications of our main results.

Keywords: Riemann Zeta function, Fox Wright- ψ-function, Generalized hypergeometric function, Hurwitz-Lerch Zeta function, Beta function.

Subject Classification: Primary 11M06, 11M35, 33B15; Secondary 11B68, 33E20, 33E30.

1. Introduction and Preliminaries

The familiar general Hurwitz-Lerch Zeta function is defined as follows Srivastava [7]:

$$\phi(z,s,a)=\sum_{l=0}^{\infty}\frac{z^l}{(l+a)^s} \tag{1.1}$$

$(a\in C/Z_0^-; s\in C$ when $|z|<1; R(s)>1$ when $|z|=1)$
The integral representation of above defined Hurwitz-Lerch Zeta function is given by (Erdelyi et al [1]p.27, Equation 1.11(3)):

$$\phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{1-ze^{-t}}\,dt \tag{1.2}$$

$Re(a)>0, Re(s)>0$ when $|z|\leq 1(z\neq 1); Re(s)>1$ when z = 1. At $a=0$, Hurwitz-Lerch Zeta function defined in (1.1) and (1.2) reduces to the Riemann Zeta function $\zeta(z, s)$ defined in Bin-Saad [5]:

$$\zeta(z, s) = \sum_{l=0}^{\infty} \frac{z^l}{(l)^s}; \zeta(z, s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{1 - ze^{-t}} dt$$

A generalization of the Hurwitz-Lerch Zeta function is also studied by Goyal and Laddha [9] as follows:

$$\phi_{\mu}^*(z, s, a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n + a)^s}$$

where $Re(\mu)$ >0, and $(\lambda)_v$ is the Pochhammer symbol with relation $(\lambda)_v = \Gamma(\lambda + v)/\Gamma(\lambda)$ and its integral representation is:

$$\phi_{\mu}^*(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{(1 - ze^{-t})^{\mu}} dt \tag{1.3}$$

where $min\{R(a), R(s)\} > 0; |z| < 1$.

The Fox-Wright generalized hypergeometric function ${}_p\psi_q^*$ which is generalization of the familiar hypergeometric function pFq defined by Erdelyi et al.[1] as:

$${}_p\psi_q^* \left[\begin{matrix} (a_1, A_1), \cdots (a_p, A_p); \\ (b_1, B_1), \cdots (b_q, B_q); \end{matrix} \; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n} \cdots (b_q)_{B_q n}} \frac{z^n}{n!} \tag{1.4}$$

At $A_i = 1 (i = 1, \cdots, p)$; $B_j = 1 (j = 1, \cdots, q)$ it reduces to generalized hypergeometric function ${}_pF_q$.

Further generalization of the above defined Hurwitz-Lerch Zeta functions $\phi_{\mu}(z, s, a)$ and $\phi_{\mu}^*(z, s, a)$ are recently studied in the following form by Garg et al [6]:

$$\phi_{\lambda,\mu,\nu}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{(\nu)_n n!} \frac{z^n}{(n + a)^s}$$

and

$$\phi_{\lambda,\mu,\nu}(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} {}_2F_1 \left[\begin{matrix} \lambda, \mu \\ \nu \end{matrix} ; ze^{-t} \right] dt \tag{1.5}$$

where $\lambda, \mu \in C, \upsilon, a \in C/Z_0; s \in C$ when $|z| < 1, Re(s + \upsilon - \lambda - \mu) > 1$ when $|z| = 1$.

Lin and Srivastava [10] also extended the Hurwitz-Lerch Zeta function in the following form:

$$\phi_{\mu,\nu}^{\rho,\sigma}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-at}{}_2\psi_1^*\left[\begin{array}{c}(\mu,\rho),(1,1)\\(\nu,\sigma)\end{array};ze^{-t}\right]dt \tag{1.6}$$

($\mu\in C;a,\nu\in C/Z_0^-;\rho,\sigma\in R^+;\rho<\sigma$ when $s,z\in C;\rho=\sigma$ and $s\in C$ when $|z|<\delta:=\rho^{-\rho}\sigma^{\sigma};\rho=\sigma$ and $Re(s-\mu+\nu)>1$ when $|z|=\delta$)

Bin-Saad [5] established the following generating functions for the Hurwitz-Lerch Zeta function defined in (1.1):

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\phi(z,s+n,a)t^n=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^{s-\lambda}(n+a-t)^{\lambda}}=V_{\lambda}(z,t;s,a) \tag{1.7}$$

where $|t|<|a|$.

In the limiting case when $t\longrightarrow t/\lambda$ and $|\lambda|\longrightarrow\infty$, (1.7) becomes

$$\sum_{0}^{\infty}\phi(z,s+n,a)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}exp\left(\frac{t}{n+a}\right)=\psi(z,t;s,a) \tag{1.8}$$

where $|t|<\infty$.

In an attempt to unify the definitions of Hurwitz-Lerch Zeta functions Srivastava [7] presented the following multiparameter Hurwitz-Lerch Zeta function as follows:

$$\phi_{(\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z,s,a)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\frac{z^n}{(n+a)^s} \tag{1.9}$$

($p,q\in N_0;\lambda_j\in C((j=1,\cdots p);a,\mu_j\in C/Z_0^-((j=1,\cdots q);\rho_j,\sigma_k\in R^+(j=1,\cdots p;k=1,\cdots,q))$)

They also obtained the following two generating relations associated with multiparameter Hurwitz-Lerch Zeta function defined in (1.9):

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\phi_{(\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z,s+n,a)t^n=\sum_{l=0}^{\infty}\frac{E_l z^l}{(l+a)^{s-\lambda}(l+a-t)^{\lambda}}=\Omega_{\lambda}(z,t;s,a) \tag{1.10}$$

where $|t| < |a|$ and $E_n = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}}$ $(n \in N_0)$

When $t \longrightarrow t/\lambda$ and $|\lambda| \longrightarrow \infty$ generating function (1.10) yields the following generating relation:

$$\sum_{n=0}^{\infty} \phi_{(\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}(z, s+n, a)\frac{t^n}{n!} = \sum_{l=0}^{\infty} \frac{E_l z^l}{(l+a)^s} exp\left(\frac{t}{l+a}\right) = \theta(z,t;s,a) \quad (1.11)$$

where $|t| < \infty$.
The truncated forms of the generating functions $\Omega_\lambda(z,t;s,a)$ and $\theta(z,t;s,a)$ are also defined by Srivastava [8]:

$$\Omega_\lambda^{(0,r)}(z,t;s,a) = \sum_{l=0}^{r} \frac{E_l z^l}{(l+a)^{s-\lambda}(l+a-t)^\lambda}; r \in N_0$$

$$\Omega_\lambda^{(r+1,\infty)}(z,t;s,a) = \sum_{l=r+1}^{\infty} \frac{E_l z^l}{(l+a)^{s-\lambda}(l+a-t)^\lambda}; r \in N_0$$

$$\theta^{(0,r)}(z,t;s,a) = \sum_{l=0}^{r} \frac{E_l z^l}{(l+a)^s} exp(\frac{t}{l+a})$$

$$\theta^{(r+1,\infty)}(z,t;s,a) = \sum_{l=r+1}^{\infty} \frac{E_l z^l}{(l+a)^s} exp(\frac{t}{l+a})$$

Which satisfy the following decomposition formulas:

$$\Omega_\lambda^{(0,r)}(z,t;s,a) + \Omega_\lambda^{(r+1,\infty)}(z,t;s,a) = \Omega_\lambda(z,t;s,a)$$
$$\theta^{(0,r)}(z,t;s,a) + \theta^{(r+1,\infty)}(z,t;s,a) = \theta(z,t;s,a) \quad (1.12)$$

The integral representation for these generating functions are defined as follows Srivastava [7] [8]:
$$\Omega_\lambda(z,w;s,a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} {}_p\psi_q^* \left[\begin{matrix} (a_1,A_1),\cdots(a_p,A_p); \\ (b_1,B_1),\cdots(b_q,B_q); \end{matrix} \; z \right] {}_1F_1(\lambda;s;wt)dt$$

where $min\{R(\text{a}), R(s)\}>0$.

$$\theta(z,w;s,a) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}{}_p\psi_q^*\left[\begin{array}{c}(a_1,A_1),\cdots(a_p,A_p);\\(b_1,B_1),\cdots(b_q,B_q);\end{array} z\right]{}_0F_1(-;s;wt)dt$$

where $min\{R(\text{a}), R(s)\}>0$.

In the recent work of Srivastava [7] [8]; these functions are widely studied in view of certain simpler functions, integrals and summation formulae. He also established some Beta integrals and their special cases in above papers.

1.1 Results Required

The following results are easily established using summation theorems for hypergeometric function due to Gauss, Watson and Whipple (see also Jaimini [3]).

$$\int_\xi^\eta (t-\xi)^\nu(\eta-t)^{\mu-1}{}_2F_1\left[\begin{array}{c}\zeta,b\\\mu\end{array};\frac{\eta-t}{\eta-\xi}\right]dt$$
$$= (\eta-\xi)^{\nu+\mu}\frac{\Gamma(\mu)\Gamma(\nu+1)\Gamma(\nu+\mu-\zeta-b+1)}{\Gamma(\nu+\mu-\zeta+1)\Gamma(\nu+\mu-b+1)} \quad (1.13)$$

where $Re(\nu) > -1, Re(\mu) > 0, Re(\nu+\mu-\zeta-b+1) > 0$.

$$\int_\xi^\eta (t-\xi)^\mu(\eta-t)^{\mu}{}_2F_1\left[\begin{array}{c}\zeta,b\\\frac{1}{2}(\zeta+b+1)\end{array};\frac{\eta-t}{\eta-\xi}\right]dt$$
$$= (\eta-\xi)^{2\mu+1}\frac{\pi\Gamma(\mu+1)\Gamma\left(\frac{\zeta+b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta-b}{2}\right)}{2^{2\mu+1}\Gamma\left(\frac{\zeta+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta}{2}\right)\Gamma\left(\mu+\frac{3-b}{2}\right)} \quad (1.14)$$

where $Re(\mu) > -1, Re(3-\zeta-b+2\lambda) > 0$.

$$\int_\xi^\eta (t-\xi)^{\mu-\nu}(\eta-t)^{\mu-1}{}_2F_1\left[\begin{array}{c}\zeta,1-\zeta\\\nu\end{array};\frac{\eta-t}{\eta-\xi}\right]dt$$
$$= (\eta-\xi)^{2\mu-\nu}\frac{\pi\Gamma(\mu)\Gamma(\nu)\Gamma(\mu-\nu+1)}{2^{2\mu-1}\Gamma\left(\frac{\nu+\zeta}{2}\right)\Gamma\left(\frac{1-\nu-\zeta}{2}\right)\Gamma\left(\mu+\frac{\zeta-\nu+1}{2}\right)\Gamma\left(\mu+\frac{2-\zeta-\nu}{2}\right)} \quad (1.15)$$

where $Re(\mu) > 0, Re(\mu-\nu+1) > 0$ and ${}_2F_1$ is the Gauss's hypergeometric function.

Motivated by the recent work due to Srivastava [7] [8]; we have established the following six results involving functions related to Hurwitz Lerch Zeta functions.

2. Main Results

For h(t)= $((t-\xi)/(\eta-\xi))^{\gamma}$ and $v(t)=(((t-\xi)(\eta-t))/(\eta-\xi)^2)^{\gamma}$ and in terms of the sequence $\{E_l\}; l \in N_0$ of the coefficients each of the following Euler Beta function integral formulas holds true:

Result-1:

$$\int_{\xi}^{\eta}(t-\xi)^{\nu}(\eta-t)^{\mu-1}{}_2F_1\left[\begin{matrix}\zeta, b\\ \mu\end{matrix};\frac{\eta-t}{\eta-\xi}\right]\Omega_{\lambda}(z, wh(t); s, a)dt$$
$$=(\eta-\xi)^{\nu+\mu}\frac{\Gamma(\mu)\Gamma(\nu+1)\Gamma(\nu+\mu-\zeta-b+1)}{\Gamma(\nu+\mu-\zeta+1)\Gamma(\nu+\mu-b+1)}$$
$$\sum_{l=0}^{\infty}\frac{E_l z^l}{(l+a)^s}{}_3\psi_2^*\left[\begin{matrix}(\lambda,1),(\nu+1,\gamma)(\nu+\mu-\zeta-b+1,\gamma)\\ (\nu+\mu-\zeta+1,\gamma),(\nu+\mu-b+1,\gamma);\end{matrix}\frac{w}{(l+a)}\right] \tag{2.1}$$

where $\eta \neq \xi; min\{R(v), R(\mu)\} > 0; \gamma > 0$.

Result-2:

$$\int_{\xi}^{\eta}(t-\xi)^{\mu}(\eta-t)^{\mu}{}_2F_1\left[\begin{matrix}\zeta, b\\ \frac{1}{2}(\zeta+b+1)\end{matrix};\frac{\eta-t}{\eta-\xi}\right]\Omega_{\lambda}(z, wv(t); s, a)dt$$
$$=(\eta-\xi)^{2\mu+1}\frac{\pi\Gamma(\mu+1)\Gamma\left(\frac{\zeta+b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta-b}{2}\right)}{2^{2\mu+1}\Gamma\left(\frac{\zeta+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta}{2}\right)\Gamma\left(\mu+\frac{3-b}{2}\right)}$$
$$\sum_{l=0}^{\infty}\frac{E_l z^l}{(l+a)^s}{}_3\psi_2^*\left[\begin{matrix}(\lambda,1),(\mu+1,\gamma)\left(\mu+\frac{3-\zeta-b}{2},\gamma\right)\\ \left(\mu+\frac{3-\zeta}{2},\gamma\right),\left(\mu+\frac{3-b}{2},\gamma\right)\end{matrix};\frac{w}{(4)^{\gamma}(l+a)}\right] \tag{2.2}$$

where $\eta \neq \xi; R(\mu) > 0; \gamma > 0$.

Result-3:

$$\int_{\xi}^{\eta}(t-\xi)^{\mu-\nu}(\eta-t)^{\mu-1}{}_2F_1\left[\begin{matrix}\zeta, 1-\zeta\\ \nu\end{matrix};\frac{\eta-t}{\eta-\xi}\right]\Omega_{\lambda}(z, wv(t); s, a)dt$$
$$=(\eta-\xi)^{2\mu-\nu}\frac{\pi\Gamma(\mu)\Gamma(\nu)\Gamma(\mu-\nu+1)}{2^{2\mu-1}\Gamma\left(\frac{\nu+\zeta}{2}\right)\Gamma\left(\frac{1-\nu-\zeta}{2}\right)\Gamma\left(\mu+\frac{\zeta-\nu+1}{2}\right)\Gamma\left(\mu+\frac{2-\zeta-\nu}{2}\right)}$$
$$\sum_{l=0}^{\infty}\frac{E_l z^l}{(l+a)^s}{}_3\psi_2^*\left[\begin{matrix}(\lambda,1),(\mu,\gamma)(\mu-\nu+1,\gamma)\\ \left(\mu+\frac{\zeta-\nu+1}{2},\gamma\right),\left(\mu+\frac{2-\zeta-\nu}{2},\gamma\right)\end{matrix};\frac{w}{(4)^{\gamma}(l+a)}\right] \tag{2.3}$$

where $\eta \neq \xi; min\{R(\mu-\upsilon), R(\mu)\} > 0; \gamma > 0$.

Result-4:

$$\int_{\xi}^{\eta}(t-\xi)^{\nu}(\eta-t)^{\mu-1}{}_2F_1\left[\begin{matrix}\zeta, b\\ \mu\end{matrix};\frac{\eta-t}{\eta-\xi}\right]\theta(z,wh(t);s,a)dt$$

$$=(\eta-\xi)^{\nu+\mu}\frac{\Gamma(\mu)\Gamma(\nu+1)\Gamma(\nu+\mu-\zeta-b+1)}{\Gamma(\nu+\mu-\zeta+1)\Gamma(\nu+\mu-b+1)}$$

$$\sum_{l=0}^{\infty}\frac{E_l z^l}{(l+a)^s}{}_2\psi_2^*\left[\begin{matrix}(\nu+1,\gamma),(\nu+\mu-\zeta-b+1,\gamma)\\ (\nu+\mu-\zeta+1,\gamma),(\nu+\mu-b+1,\gamma)\end{matrix};\frac{w}{(l+a)}\right] \tag{2.4}$$

where $\eta \neq \xi; min\{R(\nu), R(\mu)\} > 0; \gamma > 0$.

Result-5:

$$\int_{\xi}^{\eta}(t-\xi)^{\mu}(\eta-t)^{\mu}{}_2F_1\left[\begin{matrix}\zeta, b\\ \frac{1}{2}(\zeta+b+1)\end{matrix};\frac{\eta-t}{\eta-\xi}\right]\theta(z,wv(t);s,a)dt$$

$$=(\eta-\xi)^{2\mu+1}\frac{\pi\Gamma(\mu+1)\Gamma\left(\frac{\zeta+b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta-b}{2}\right)}{2^{2\mu+1}\Gamma\left(\frac{\zeta+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta}{2}\right)\Gamma\left(\mu+\frac{3-b}{2}\right)}$$

$$\sum_{l=0}^{\infty}\frac{E_l z^l}{(l+a)^s}{}_2\psi_2^*\left[\begin{matrix}(\mu+1,\gamma),\left(\mu+\frac{3-\zeta-b}{2},\gamma\right)\\ \left(\mu+\frac{3-\zeta}{2},\gamma\right),\left(\mu+\frac{3-b}{2},\gamma\right)\end{matrix};\frac{w}{(4)^{\gamma}(l+a)}\right] \tag{2.5}$$

where $\eta \neq \xi; R(\mu) > 0; \gamma > 0$.

Result-6:

$$\int_{\xi}^{\eta}(t-\xi)^{\mu-\nu}(\eta-t)^{\mu-1}{}_2F_1\left[\begin{matrix}\zeta, 1-\zeta\\ \nu\end{matrix};\frac{\eta-t}{\eta-\xi}\right]\theta(z,wv(t);s,a)dt$$

$$=(\eta-\xi)^{2\mu-\nu}\frac{\pi\Gamma(\mu)\Gamma(\nu)\Gamma(\mu-\nu+1)}{2^{2\mu-1}\Gamma\left(\frac{\nu+\zeta}{2}\right)\Gamma\left(\frac{1-\nu-\zeta}{2}\right)\Gamma\left(\mu+\frac{\zeta-\nu+1}{2}\right)\Gamma\left(\mu+\frac{2-\zeta-\nu}{2}\right)}$$

$$\sum_{l=0}^{\infty}\frac{E_l z^l}{(l+a)^s}{}_2\psi_2^*\left[\begin{matrix}(\mu,\gamma),(\mu-\nu+1,\gamma)\\ \left(\mu+\frac{\zeta-\nu+1}{2},\gamma\right),\left(\mu+\frac{2-\zeta-\nu}{2},\gamma\right)\end{matrix};\frac{w}{(4)^{\gamma}(l+a)}\right] \tag{2.6}$$

where $\eta \neq \xi; min\{R(\mu-\nu), R(\mu)\} > 0; \gamma > 0$.

Provided that both sides of each of the assertions (2.1) to (2.6) exists.

Outline of Proofs

Proof of (2.1): For convenience, we denote left hand side of assertion (2.1) by I_1

and upon using definition of generating function Ω_λ (z,t;s,a) given in (1.10), we get

$$I_1 = \int_\xi^\eta (t-\xi)^\nu (\eta-t)^{\mu-1} {}_2F_1 \left[\begin{matrix} \zeta, b \\ \mu \end{matrix} ; \frac{\eta-t}{\eta-\xi} \right] \sum_{l=0}^{\infty} \frac{E_l z^l}{(l+a)^{s-\lambda} \left[l+a-w \left(\frac{t-\xi}{\eta-\xi} \right)^\gamma \right]^\lambda} dt$$

Now in view of binomial expansion and on changing the order of integration and summation, it takes the following form :

$$I_1 = \sum_{l,n=0}^{\infty} \frac{E_l z^l (\lambda)_n w^n}{n!(l+a)^{s+n} (\eta-\xi)^{n\gamma}} \int_\xi^\eta (t-\xi)^{\nu+\gamma n} (\eta-t)^{\mu-1} {}_2F_1 \left[\begin{matrix} \zeta, b \\ \mu \end{matrix} ; \frac{\eta-t}{\eta-\xi} \right] dt$$

On evaluating the inner integral using the result (1.13) then in view of the relation $(\lambda)_\nu = (\Gamma(\lambda+\nu))/(\Gamma(\lambda))$ and interpreting the series with w-variable with the help of (1.4) we at once arrive at the desired result in (2.1).

The results in (2.2) and (2.3) are proved on following similar lines in view of (1.10) and using the results in (1.14) and (1.15) respectively therein.

Proof of (2.4): For convenience, we denote the left hand side of assertion (2.4) by Δ and upon using definition of generating function θ(z,t;s,a) given in (1.11), we get

$$\Delta = \int_\xi^\eta (t-\xi)^\nu (\eta-t)^{\mu-1} {}_2F_1 \left[\begin{matrix} \zeta, b \\ \mu \end{matrix} ; \frac{\eta-t}{\eta-\xi} \right] \sum_{l=0}^{\infty} \frac{E_l z^l}{(l+a)^s} exp \left[\frac{w}{(a+l)} \left(\frac{t-\xi}{\eta-\xi} \right)^\gamma \right] dt$$

Now on expending exponential function and then changing the order of integration and summation, it takes the following form

$$\Delta = \sum_{l,n=0}^{\infty} \frac{E_l z^l w^n}{n!(l+a)^{s+n} (\eta-\xi)^{n\gamma}} \int_\xi^\eta (t-\xi)^{\nu+\gamma n} (\eta-t)^{\mu-1} {}_2F_1 \left[\begin{matrix} \zeta, b \\ \mu \end{matrix} ; \frac{\eta-t}{\eta-\xi} \right] dt$$

We evaluate the inner integral using the result in (1.13) then in view of the relation $(\lambda)_v = (\Gamma(\lambda+v))/(\Gamma(\lambda))$ and interpreting the series with w-variable with the help of (1.4) we at once arrive at the desired result in (2.4). The results in (2.5) and (2.6) are proved on following similar lines in view of (1.11) and using the results in (1.14) and (1.15) respectively therein.

3. Applications

The main results established in Section-2 are very general in nature; therefore as applications of these results many known and new results can be obtained. Some of these are shown to be reduced from our main results.

(1) If in (2.1), we take $\xi = 0, \eta = 1$ this reduces to the following Beta integral:

$$\int_0^1 t^{\nu}(1-t)^{\mu-1} {}_2F_1\left[\begin{matrix}\zeta, b \\ \mu\end{matrix} ; 1-t\right] \Omega_{\lambda}(z, wh(t); s, a)dt$$

$$= \frac{\Gamma(\mu)\Gamma(\nu+1)\Gamma(\nu+\mu-\zeta-b+1)}{\Gamma(\nu+\mu-\zeta+1)\Gamma(\nu+\mu-b+1)}$$

$$\sum_{l=0}^{\infty} \frac{E_l z^l}{(l+a)^s} {}_3\psi_2^* \left[\begin{matrix}(\lambda,1),(\nu+1,\gamma)(\nu+\mu-\zeta-b+1,\gamma) \\ (\nu+\mu-\zeta+1,\gamma),(\nu+\mu-b+1,\gamma)\end{matrix} ; \frac{w}{(l+a)}\right] \tag{3.1}$$

(2) If in (2.3) and (2.5), we take p = q = 0, then these reduce to the following results involving generating functions $V_{\lambda}(z,t;s,a)$ and $\psi(z,t;s,a)$ defined in (1.7) and (1.8) respectively:

$$\int_{\xi}^{\eta} (t-\xi)^{\mu-\nu}(\eta-t)^{\mu-1} {}_2F_1\left[\begin{matrix}\zeta, 1-\zeta \\ \nu\end{matrix} ; \frac{\eta-t}{\eta-\xi}\right] V_{\lambda}(z, wv(t); s, a)dt$$

$$= (\eta-\xi)^{2\mu-\nu} \frac{\pi\Gamma(\mu)\Gamma(\nu)\Gamma(\mu-\nu+1)}{2^{2\mu-1}\Gamma\left(\frac{\nu+\zeta}{2}\right)\Gamma\left(\frac{1-\nu-\zeta}{2}\right)\Gamma\left(\mu+\frac{\zeta-\nu+1}{2}\right)\Gamma\left(\mu+\frac{2-\zeta-\nu}{2}\right)}$$

$$\sum_{l=0}^{\infty} \frac{z^l}{(l+a)^s} {}_3\psi_2^* \left[\begin{matrix}(\lambda,1),(\mu,\gamma)(\mu-\nu+1,\gamma) \\ \left(\mu+\frac{\zeta-\nu+1}{2},\gamma\right),\left(\mu+\frac{2-\zeta-\nu}{2},\gamma\right)\end{matrix} ; \frac{w}{(4)^{\gamma}(l+a)}\right] \tag{3.2}$$

$$\int_{\xi}^{\eta} (t-\xi)^{\mu}(\eta-t)^{\mu} {}_2F_1\left[\begin{matrix}\zeta, b \\ \frac{1}{2}(\zeta+b+1)\end{matrix} ; \frac{\eta-t}{\eta-\xi}\right] \psi(z, wv(t); s, a)dt$$

$$= (\eta-\xi)^{2\mu+1} \frac{\pi\Gamma(\mu+1)\Gamma\left(\frac{\zeta+b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta-b}{2}\right)}{2^{2\mu+1}\Gamma\left(\frac{\zeta+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta}{2}\right)\Gamma\left(\mu+\frac{3-b}{2}\right)}$$

$$\sum_{l=0}^{\infty} \frac{z^l}{(l+a)^s} {}_2\psi_2^* \left[\begin{matrix}(\mu+1,\gamma),\left(\mu+\frac{3-\zeta-b}{2},\gamma\right) \\ \left(\mu+\frac{3-\zeta}{2},\gamma\right),\left(\mu+\frac{3-b}{2},\gamma\right)\end{matrix} ; \frac{w}{(4)^{\gamma}(l+a)}\right] \tag{3.3}$$

where $\eta \neq \xi; R(\mu) > 0; \gamma > 0$.

It is worthwhile to mention here that at p = q = 0, $\lambda = 0$, the results in (2.1), (2.2), (2.3) reduce to the corresponding results involving the Hurwitz-Lerch Zeta functions defined in (1.1).

Conclusion:
The results established and studied in this paper are very general Euler type integrals involving certain class of functions related to Hurwitz-Lerch Zeta functions. So many integrals for simpler functions defined in [6], [9] and [10] can be obtained which are very useful in transform calculus.

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