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A note on Stieltjes Transform

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Dedicated to Prof. Hari M. Srivastava on his 75^{th} ***birth anniversary***

Abstract: In this paper an attempt has been made to discuss about the Stieltjes transform, its properties and its generalization.

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1. Introduction

The Stieltjes transform comes out naturally by repeated applications of Laplace transform. If

$$g(s) = \int_0^\infty e^{-su}\Phi(u)du,$$

where

$$\Phi(u) = \int_0^\infty e^{-ut}g(t)dt$$

Then

$$\begin{aligned} g(s) &= \int_0^\infty e^{-su}\left\{\int_0^\infty e^{-ut}g(t)dt\right\}du \\ &= \int_0^\infty g(t)\left\{\int_0^\infty e^{-(s+t)u}du\right\}dt \\ &= \int_0^\infty \left[\frac{e^{-(s+t)u}}{(s+t)}\right]_0^\infty g(t)dt \end{aligned}$$

$$= \int_0^\infty \frac{g(t)}{s+t} dt. \tag{1.1}$$

(1.1) is the special case of the general Stieltjes transform which is defined as,

$$g_a(s) = \int_0^\infty \frac{g(t)}{(s+t)^a} dt, \tag{1.2}$$

provided the integral exists. a is an arbitrary complex number and g_a is an analytic function on some region of the s-plane. This property of $g_a(s)$ is sufficient to guarantee the existence of both the multiple-integral representation and the contour integral representation of the generalized Stieltjes transform.

We shall make use the following functions.

Gaussian hypergeometric function,

$${}_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}, \tag{1.3}$$

where $(a)_0 = 1,\ (a)_n = a(a+1)...(a+n-1)$. For the convergence of the series in (1.3), $|z| < 1$ is needed.
Confluent hypergeometric series is defined as,

$${}_1F_1[a; b; z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}, \tag{1.4}$$

where for convergence $|z| < \infty$ i.e. it is convergent in the whole complex plane.

$$R(a; b, b'; x, y) = y^{-a}\ {}_2F_1 \left[a, b; b+b'; 1 - \frac{x}{y} \right]. \tag{1.5}$$

$$S(b, b'; x, y) = e^y\ {}_1F_1[b; b+b'; x-y]. \tag{1.6}$$

Also,

$${}_2F_1[a, b; c; x] = R(a; b, c-b; 1-x, 1), \tag{1.7}$$

$${}_1F_1[a; b; x] = S(a, b-a; x, 0), \tag{1.8}$$

$$R(a, b; \lambda z) = \lambda^{-a} R(a, b, z), \tag{1.9}$$

$$S(b, \lambda + z) = e^{\lambda} S(b, z). \tag{1.10}$$

$${}_1F_0[a; -; z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} = (1-z)^{-a} \tag{1.11}$$

2. Different representation of Stieltjes transform

Since,

$$ {}_1F_0[a; -; -z] = (1+z)^{-a} \tag{2.1}$$

so, if we take $z = \dfrac{t}{s}$ then

$$ {}_1F_0\left[a; -; -\frac{t}{s}\right] = \left(1+\frac{t}{s}\right)^{-a} = (s+t)^{-a}s^a. \tag{2.2}$$

From (2.2) we have

$$ (s+t)^{-a} = \frac{1}{(s+t)^a} = s^{-a}\ {}_1F_0\left[a; -; -\frac{t}{s}\right]. \tag{2.3}$$

Stieltjes transform

$$ g_a(s) = \int_0^\infty \frac{g(t)}{(s+t)^a}dt $$

can be written in the form,

$$ g_a(s) = \frac{1}{s^a}\int_0^\infty {}_1F_0\left[a; -; -\frac{t}{s}\right] g(t)dt. \tag{2.4}$$

From (1.8) we have,

$$ {}_1F_1[a; b; x] = S(a, b-a; x, 0) \tag{2.5}$$

Taking $b = 0, x = -\dfrac{t}{s}$ in (2.5) we have

$$ {}_1F_0\left[a; -; -\frac{t}{s}\right] = S\left(a, -a; -\frac{t}{s}, 0\right). \tag{2.6}$$

Putting the value of ${}_1F_0$ from (2.6) in (2.4) we get another form of Stieltjes transform,

$$ g_a(s) = \frac{1}{s^a}\int_0^\infty S\left(a, -a; -\frac{t}{s}, 0\right) g(t)dt. \tag{2.7}$$

(2.4) can also be expressed as,

$$ g_a(s) = \frac{1}{s^a}\int_0^\infty {}_2F_1\left[a, b; b; -\frac{t}{s}\right] g(t)dt \tag{2.8}$$

Making use of (1.7) we can rewrite (2.8) as,

$$g_a(s) = \frac{1}{s^a}\int_0^\infty R\left(b; a, c-a; 1+\frac{t}{s}, 1\right) g(t)dt. \tag{2.9}$$

3. Basic (q-) generalization of Stieltjes Transform

The basic hypergeometric series is defined by,

$${}_2\Phi_1(a, b; c; q; z) = {}_2\Phi_1\left[\begin{matrix} a, b; q; z \\ c \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n, \tag{3.1}$$

where $(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}),\ n = 1, 2, 3, ...$
${}_2\Phi_1$ given in (3.1) is the basic analogue of ${}_2F_1$ given in (1.3). Taking $b = c$ in (3.1), it yields,

$${}_1\Phi_0(a; -; q; z) = \sum_{n=0}^{\infty}\frac{(a;q)_n z^n}{(q;q)_n}. \tag{3.2}$$

${}_1\Phi_0$ defined in (3.2) is the basic analogue of ${}_1F_0$ given in (1.11). Basic binomial theorem is given as,

$${}_1\Phi_0(a; -; q; z) = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{3.3}$$

The identity (3.3) is the basic analogue of the binomial theorem

$${}_1F_0(a; -; z) = (1-z)^{-a}. \tag{3.4}$$

With all these definition, the basic or q-generalization of the Stieltjes transform is given by,

$$g_{q,a}(s) = \frac{1}{s^a}\int_0^\infty {}_1\Phi_0\left(a; -; q; -\frac{t}{s}\right) g(t)d_qt. \tag{3.5}$$

The integral given in (3.5) is the basic or q-integral.
Making use of (3.3) we have,

$$g_{q,a}(s) = \frac{1}{s^a}\int_0^\infty \frac{\left(-\frac{at}{s}; q\right)_\infty}{\left(-\frac{t}{s}; q\right)_\infty} g(t)d_qt, \tag{3.6}$$

where q-integral is defined as [Gasper and Rahman: 1;(1.11.4) p. 19]

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n. \tag{3.7}$$

Making use of (3.7) we can write (3.6) as,

$$g_{q,a}(s) = \left(\frac{1-q}{s^a}\right) \sum_{n=-\infty}^{\infty} \frac{\left(-\frac{aq^n}{s};q\right)_\infty}{\left(-\frac{q^n}{s};q\right)_\infty} g(q^n)q^n. \tag{3.8}$$

4. Generalized Stieltjes Transform

In the Stieltjes Transform,

$$g_a(s) = \int_0^\infty \frac{g(t)}{(s+t)^a} dt,$$

$g_a(s)$ is analytic in the complex-plane having a cut along the non-positive real axis. Let D be the s-plane having cut along the non-positive real axis. If $s_1, s_2, ..., s_k$ are points in D and $c = \sum b_i$ such that $c \neq 0, -1, -2, ...$, then the generalized Stieltjes transform is given by,

$$G_a(b,s) = \int_0^\infty R(a,b,s+t)g(t)dt, \tag{4.1}$$

where R is given by (1.9).

Stieltjes transform has not been studied as extensively as the Laplace or Fourier transform, perhaps because the Kernel $(s+t)^{-a}$ is not an entire function of s.

(a) If we take $g(t) = t^{\nu-1}$ then its ordinary Stieltjes transform is given as,

$$\int_0^\infty t^{\nu-1}(s+t)^{-a}dt = \frac{\Gamma(\nu)\Gamma(a-\nu)}{\Gamma(a)s^{a-\nu}}, \tag{4.2}$$

provided $Re(a) > Re(\nu) > 0$.

The generalized Stieltjes transform of $t^{\nu-1}$ is

$$\int_0^\infty t^{\nu-1}R(a,b,s+t)dt = \frac{\Gamma(\nu)\Gamma(a-\nu)}{\Gamma(a)} R(a-\nu,b,s), \tag{4.3}$$

where $Re(a) > Re(\nu) > 0$.

Similar other results can also be established. For details about the Stieltjes transform one is referred [2].

References

[1] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Cambridge University Press (1991).

[2] Murray Rognlie, D., Generalized integral transforms, Ph.D. Thesis submitted to Iowa State University, Ames, Iowa (1969)