J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.4, No.2 (2015), pp. 71-76

Higher order derivative on meromorphic functions in terms of subordination

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Dedicated to Prof. Hari M. Srivastava on his 75^{th} ***birth anniversary***

Abstract: This paper investigates sharp coefficient bounds, integral representation, extreme point and operator properties of a certain class associated with functions which are meromorphic in the punctured unit disk.
keywords and phrases: Meromorphic function, Coefficient estimate, Extreme point, Integral representation, Integral operator.

2000 Mathematics Subject Classification : 30C45, 30C50.

1. Introduction

Let $\sum$ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=3}^{\infty} a_n z^n \quad (a_n \geq 0, \quad n \in \mathbb{N}), \tag{1.1}$$

which are analytic and univalent in the punctured unit disk $\Delta^* = \{z : \quad 0 < |z| < 1\}$.
For $f(z) \in \sum$, Ghanim and Darus [3] were defined a linear operator $I^k \quad (k = 0, 1, 2, ...)$ as follows:

$$\begin{array}{l} I^0 f(z) = f(z) \\ I^k f(z) = z(I^{k-1} f(z))' + \frac{2}{z} = \frac{1}{z} + \sum_{n=3}^{+\infty} n^k a_n z^n. \end{array} \tag{1.2}$$

For $A = B + (C - B)(1 - D)$, $-1 \leq B < C \leq 1$ and $0 \leq D < 1$, we let $\sum_{A,B}^{K}$ consists of function $f \in \sum$ satisfying the condition

$$-\frac{zF^{(4)}(z)}{F'''(z)} < 4\frac{1 + Az}{1 + Bz}, \tag{1.3}$$

where $F(z) = I^k f(z)$ is defined by (1.2).
For other subclass of meromorphic univalent functions, we can see the recent works of many authors in [1] and [2].

2. Main results

In this section we find sharp coefficient bounds and Integral representation for the class $\sum_{A,B}^{k}$.

Theorem 2.1. *Let $f(z) \in \sum$, then $f(z) \in \sum_{A,B}^{k}$ if and only if*

$$\sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))]a_n < 24(C-B)(1-D). \quad (2.1)$$

The result is sharp for the function $G(z)$ given by (2.2)

$$G(z) = \frac{1}{z} + \frac{24(C-B)(1-D)}{n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))]} z^n \quad (n = 3, 4, ...)$$

Proof. *Let $f(z) \in \sum_{A,B}^{k}$, then the inequality (1.3) or equivalently*

$$\left| \frac{zF^{(4)}(z) + 4F'''(z)}{zBF^{(4)}(z) + 4[B+(C-B)(1-D)]F'''(z)} \right| < 1,$$

holds true, therefore by making use of (1.2) we have

$$\left| \frac{\sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)^2 a_n z^{n-3}}{-24(C-B)(1-D)z^{-4} + \sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)[n-3+4(B+(C-B)(1-D))]a_n z^{n-3}} \right| < 1.$$

Since $Rez \leq |z|$ for all z, thus

$$Re\left\{ \frac{\sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)^2 a_n z^{n-3}}{-24(C-B)(1-D)z^{-4} + \sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)[n-3+4(B+(C-B)(1-D))]a_n z^{n-3}} \right\} < 1.$$

By letting $z \to 1$ through real values, we get the required result.
Conversely, let (2.1) holds true. If we let $z \in \partial\triangle^*$, where $\partial\triangle^*$ denotes the boundary of $\triangle^*$, then we have

$$\left| \frac{zF^{(4)}(z) + 4F'''(z)}{zBF^{(4)}(z) + 4[B+(C-B)(1-D)]F'''(z)} \right|$$

$$\leq \frac{\sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)^2|a_n|}{24(C-B)(1-D) - \sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)[n-3+4(B+(C-B)(1-D))]|a_n|} < 1.$$

Thus by the maximum modolus Theorem we conclude $f(z) \in \sum_{A,B}^{k}$.
Now the proof is complete.

Theorem 2.2. *If* $f(z) \in \sum_{A,B}^{k}$, *then*

$$f(z) = \int_0^z \int_0^z \int_0^z exp \int_0^z \frac{4(AW(z)-1)}{z(1-BW(z))} d\alpha d\beta d\gamma d\theta$$

where $A = B + (C-D)(1-D)$ *and* $|W(z)| < 1$.

Proof. *Since* $f(z) \in \sum_{A,B}^{k}$, *so (1.3) holds true or equivalently we have*

$$\left| \frac{zF^{(4)}(z) + 4F'''(z)}{zBF^{(4)}(z) + 4AF'''(z)} \right| < 1,$$

where $A = B + (C-D)(1-D)$. *Hence*

$$\frac{zF^{(4)}(z) + 4F'''(z)}{zBF^{(4)}(z) + 4AF'''(z)} = W(z),$$

where $|W(z)| < 1$, $z \in \Delta^*$. *This yields*

$$\frac{F^4(z)}{F'''(z)} = \frac{4(AW(z)) - 1}{z(1-BW(z))}.$$

after four times integration we obtain the required result.

Remark. Theorem 2.1 shows that if $f(z) \in \sum_{A,B}^{k}$, then

$$|a_n| \leq \frac{24(C-B)(1-D)}{2 \times 3^{k+1}(1+4A)}, \quad n = 3, 4, ... \tag{2.3}$$

where $A = B + (C-B)(1-D)$.

3. Extreme points and convex linear combination

Our next theorems involve extreme points and convex linear combination property.

Theorem 3.1. *The function* $f(z)$ *of the form (1.1) belongs to* $\sum_{A,B}^{k}$ *if and only if it can be expressed by*

$$f(z) = \sum_{n=2}^{+\infty} d_n f_n(z) \quad d_n \geq 0,$$

where $f_2(z) = z^{-1}$,

$$f_n(z) = z^{-1} + \frac{24(C-B)(1-D)}{n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))]} z^n$$

$$(n = 3, 4, ... \text{ and } \quad \sum_{n=2}^{+\infty} d_n = 1)$$

Proof. *Let* $f(z) = \sum_{n=2}^{+\infty} d_n f_n(z)$

$$= d_2 f_2(z) + \sum_{n=2}^{+\infty} d_n \left[z^{-1} + \frac{24(C-B)(1-D)}{n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))]} z^n \right]$$

$$= z^{-1} + \sum_{n=3}^{+\infty} \frac{24(C-B)(1-D)}{n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))]} d_n z^n.$$

Now by Theorem 2.1 we conclude that $f(z) \in \sum_{A,B}^{k}$.
Conversely if $f(z)$ *is given by (1.1) belongs to* $\sum_{A,B}^{k}$, *by letting* $d_2 = 1 - \sum_{n=3}^{+\infty} d_n$ *where*

$$d_n = \frac{n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))]}{24(C-B)(1-D)} a_n \quad n = 3, 4, ...$$

we conclude the required result.

Theorem 3.2. *The class* $\sum_{A,B}^{k}$ *is closed under convex linear combination.*

Proof. *Suppose that the function* $f_1(z)$ *and* $f_2(z)$ *defined by*

$$f_j(z) = \frac{1}{z} + \sum_{n=3}^{+\infty} a_{n,j} z^n \qquad j = 1, 2, \ z \in \Delta^*$$

are in the class $\sum_{A,B}^{k}$. *Setting*

$$f(z) = \eta f_1(z) + (1-\eta) f_2(z), \quad (0 \leq \eta < 1)$$

we obtain

$$f(z) = \frac{1}{z} + \sum_{n=3}^{+\infty} (\eta a_{n,1} + (1-\eta) a_{n,2}) z^n.$$

In the view of Theorem 2.1, we have

$\sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))](\eta a_{n,1}+(1-\eta)a_{n,2})$

$= \eta \sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))]a_{n,1}$

$+(1-\eta)\sum_{n=3}^{+\infty} n^{k+1}(n-1)(n-2)[2n-5+4(B+(C-B)(1-D))]a_{n,2}$

$< \eta[24(C-B)(1-D)] + (1-\eta)[24(C-B)(1-D)]$

$= 24(C-B)(1-D),$

which completes the proof.

4. Special operators

The main objective of this section is to define two operators on the functions $f \in \sum_{A,B}^{k}$.
Furthermore, we verify properties of these operators.
For $f \in \sum_{A,B}^{k}$ we define

(1) $\tau^{\gamma}(f(z)) = \gamma \int_0^1 u^{\gamma} f(uz)du, \quad \gamma > 1$
(2) $L^*(a,c)f(z) = \tilde{\phi}(a,c;z) * f(z)$, where

$$\tilde{\phi}(a,c;z) = \frac{1}{z} + \sum_{n=3}^{+\infty} \left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| a_n z^n, \quad c \neq 0,-1,-2,... \quad , a \in \mathbb{C} - \{0\},$$

$(x)_n$ is the pochhammer symbol and " $*$ " denotes the Hadamard product.
We note that $\tilde{\phi}(a,c;z) = \frac{1}{z}\,{}_2F_1(1,a,c;z)$ where

$${}_2F_1(b,a,c;z) = \sum_{n=0}^{+\infty} \frac{(b)_n(a)_n}{(c)_n}\frac{z^n}{n!}$$

is the well-known Gaussian hypergeometric function, see [4].

Theorem 4.1. *If $f \in \sum_{A,B}^{k}$ then $\tau^{\gamma}(f(z))$ and $L^*(a,c)f(z)$ are also in the same class.*

Proof. *By a simple calculation we conclude that*

$$\tau^{\gamma}(f(z)) = \frac{1}{z} + \sum_{n=3}^{+\infty} \frac{\gamma}{\gamma+1+n} a_n z^n$$

and since $\frac{\gamma}{\gamma+1+n}$, *by theorem 2.1 we conclude the required result.*
Also by using Hadamdard product we obtain

$$L^*(a,c)f(z) = \frac{1}{z} + \sum_{n=3}^{+\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n z^n,$$

and we easily conclude the result.

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