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On certain transformations of basic hypergeometric functions

G.S. Pant, K.B. Chand and V.P. Pande*
Department of Mathematics,
S.S.J. College, Almora (U.K.) India
*Prof & Head, Department of Mathematics,
S.S.J. College, Almora (U.K.) India
E-mail:- gspant2070@rediffmail.com

Abstract: In this paper we shall attempt to establish certain interesting transformation of basic hypergeometric functions by exploiting certain known summation formulae.
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1. Notations

For any numbers a and q, real or complex and $|q| < 1$, let

$$[\alpha; q]_n \equiv [\alpha]_n = \begin{cases} (1-\alpha)(1-\alpha q)(1-\alpha q^2)...(1-\alpha q^{n-1}); & n > 0 \\ 1 & ; \quad n = 0 \end{cases} \tag{1.1}$$

Accordingly, we have

$$[\alpha; q]_\infty = \prod_{n=0}^{\infty}(1-\alpha q^n)$$

Also,

$$[a_1, a_2, a_3, ..., a_r; q]_n \equiv [a_1; q]_n [a_2; q]_n [a_3; q]_n ... [a_r; q]_n. \tag{1.2}$$

Now, we define a basic hypergeometric function

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a_1; q]_n [a_2; q]_n ... [a_r; q]_n z^n}{[q; q]_n [b_1; q]_n [b_2; q]_n ... [b_s; q]_n} \quad . \tag{1.3}$$

valid for $|q| < 1, \quad \lambda > 0$ and if $\lambda = 0$ then for $|z| < 1$.

Other notations and definitions appearing in this paper shall stand for their usual meanings.

2. Introduction

Andrews and Askey [1] applied the following transformation

$$ {}_2\Phi_1\left[\begin{matrix} a^2, b; q; x \\ c \end{matrix}\right] = \frac{[a^2, bx; q]_\infty}{[c, x; q]_\infty}\, {}_2\Phi_1\left[\begin{matrix} c/a^2, x; q; a^2 \\ bx \end{matrix}\right] \tag{2.1} $$

for obtaining the following summation formula

$$ {}_2\Phi_1\left[\begin{matrix} b^2, b^2/c; q^2; cq/b^2 \\ c \end{matrix}\right] = \frac{1}{2}\frac{[q, b^2; q^2]_\infty}{[c, cq/b^2; q^2]_\infty}\left\{\frac{[c/b; q]_\infty}{[b; q]_\infty} + \frac{[-c/b; q]_\infty}{[-b; q]_\infty}\right\} \tag{2.2} $$

and applied it to establish certain identities of Rogers-Ramanujan type.
Also, Verma and Jain [3] exploited (2.1) to establish the following summations,

$$ {}_2\Phi_1\left[\begin{matrix} a^2, b; q; \frac{\sqrt{q}}{b} \\ a^2q/b \end{matrix}\right] = \frac{1}{2}\frac{[a^2, \sqrt{q}; q]_\infty}{[a^2q/b, \sqrt{q}/b; q]_\infty}\left\{\frac{[a\sqrt{q}/b; \sqrt{q}]_\infty}{[a; \sqrt{q}]_\infty} + \frac{[-a\sqrt{q}/b; \sqrt{q}]_\infty}{[-a; \sqrt{q}]_\infty}\right\}, \tag{2.3} $$

Verma and Jain [3; (3.5), p. 74]

$$ {}_2\Phi_1\left[\begin{matrix} a^2, b; q; \frac{q^{3/2}}{b} \\ a^2q/b \end{matrix}\right] $$

$$ = \frac{[a^2, \sqrt{q}; q]_\infty}{2a[a^2q/b, \sqrt{q}/b; q]_\infty}\left\{\frac{[a\sqrt{q}/b; \sqrt{q}]_\infty}{[a; \sqrt{q}]_\infty} + \frac{[-a\sqrt{q}/b; \sqrt{q}]_\infty}{[-a; \sqrt{q}]_\infty}\right\}, \tag{2.4} $$

Verma and Jain [3; (3.6), p. 75]

and

$$ {}_2\Phi_1\left[\begin{matrix} a^2, b; q; q^{3/2}/b \\ a^2q^2/b \end{matrix}\right] $$

$$ = \frac{[a^2, \sqrt{q}, q^2/b; q]_\infty}{2a[a^2q^2/b, q/b, q^{3/2}/b; q]_\infty}\left\{\frac{[aq/b; \sqrt{q}]_\infty}{[a; \sqrt{q}]_\infty} - \frac{[-aq/b; \sqrt{q}]_\infty}{[-a; \sqrt{q}]_\infty}\right\} \tag{2.5} $$

Verma and Jain [3; (3.7), p. 75]

We find that the above results could be very helpful in establishing interesting transformations which otherwise do not look possible. In this paper we shall use

(2.3)- (2.5) and the following known summations due to Verma and Jain [3] to establish our transformations,

$$ {}_4\Phi_3\left[\begin{array}{l} q^{-n}, x^2y^2q^{n+1}, x, -xq; q; q \\ xyq, -xyq, x^2q \end{array}\right] = \frac{x^n[q;q]_n[x^2q^2;q^2]_m[y^2q^2;q^2]_m}{[x^2y;q]_n[x^2y^2q^2;q^2]_m[q^2;q^2]_m} \tag{2.6} $$

where m is the greatest integer $\leq n/2$,

Verma and Jain [4; (2.25), p. 1028]

$$ {}_4\Phi_3\left[\begin{array}{l} b^2x^4q^{2+2n}, x^2, x^2q, q^{-2n}; q^2; q^2 \\ bx^2q, bx^2q^2, x^4q^2 \end{array}\right] = \frac{x^{2n}[-q;q]_n[bq;q]_n}{[-x^2q;q]_n[bx^2q;q]_n} \tag{2.7} $$

Verma and Jain [4; (2.32), p. 1029]

$$ {}_4\Phi_3\left[\begin{array}{l} x, -xq, bx^2q^{2+n}, q^{-n}; q; q \\ x^2q^2, xq\sqrt{b}, -xq\sqrt{b} \end{array}\right] $$
$$ = \frac{x^n[q;q]_n[bxq^2;q]_n[bx^2q^3;q^2]_m[bq^2;q^2]_m[xq^2;q]_{2m}}{[xq;q]_n[bx^2q^2;q]_n[q^2;q^2]_m[x^2q^3;q^2]_m[bxq^2;q]_{2m}} \tag{2.8} $$

where m is the greatest integer $\leq n/2$,

Verma and Jain [4; (3.2), p. 1033]

$$ {}_5\Phi_4\left[\begin{array}{l} x, aq^{1+n}, \sqrt{x}/q, -\sqrt{x}/q, q^{-n}; q; q \\ \sqrt{aq}, -\sqrt{aq}, x/q, xq \end{array}\right] $$
$$ = \frac{x^{n-m}[aq/x;q]_n[q;q]_n[aq^2;q^2]_m[xq;q^2]_m}{q^m[aq;q]_n[xq;q]_n[q^2;q^2]_m[aq/x;q^2]_m} \tag{2.9} $$

where m is the greatest integer $\leq n/2$,

Verma and Jain [4; (3.5), p. 1033]

$$ {}_5\Phi_4\left[\begin{array}{l} a, aq, aq^2, a^3q^{3+3n}, q^{-3n}; q^3; q^3 \\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^2, -a^{3/2}q^2 \end{array}\right] = \frac{a^n[q^3;q^3]_n[aq;q]_n}{[a^3q^3;q^3]_n[q;q]_n}, \tag{2.10} $$

Verma and Jain [4; (4.2), p. 1035]

$$ {}_5\Phi_4\left[\begin{array}{l} x, \omega xq, \omega^2xq, x^3q^{n+4}, q^{-n}; q; q \\ (xq)^{3/2}, -(xq)^{3/2}, x^{3/2}q^2, -x^{3/2}q^2 \end{array}\right] $$
$$ = \frac{x^n[x^2q^4;q]_n[q;q]_n[x^3q^6;q^3]_m[xq^3;q]_{3m}}{[x^3q^4;q]_n[xq;q]_n[q^3;q^3]_m[x^2q^4;q]_{3m}}, \tag{2.11} $$

Verma and Jain [4; (4.4), p. 1036]

$$ {}_5\Phi_4\left[\begin{array}{l} a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}, aq^{1+n}, q^{-n}; q; q \\ q\sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{array}\right] = \frac{a^{(n-m)/2}[q;q]_n[aq^3;q^3]_m}{[aq;q]_n[q^3;q^3]_m}, \qquad (2.12) $$

Verma and Jain [4; (4.5), p. 1036]

and

$$ {}_6\Phi_5\left[\begin{array}{l} a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}, q\sqrt{a}, aq^{1+n}, q^{n}; q; q \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a} \end{array}\right] $$

$$ = \frac{[q;q]_n[\sqrt{a};q]_n[aq^3;q^3]_m[q^6\sqrt{a};q^3]_m(a)^{(n-m)/2}}{[aq;q]_n[q^2\sqrt{a};q]_n[q^3;q^3]_m[\sqrt{a};q^3]_m} \qquad (2.13) $$

where m is the greatest integer $\leq n/2$,

Verma and Jain [4; (4.8), p. 1037]

We shall also use the following well known series identity,

$$ \sum_{n=0}^{\infty}\sum_{r=0}^{n} A(n,r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A(n+r,r) \qquad (2.14) $$

3. Transformations

In this section we shall establish our transformations.

(i) In the first place we prove the following,

$$ \sum_{n=0}^{\infty}\Omega_n z^n \frac{x^n[q;q]_n[x^2q^2;q^2]_m[y^2q^2;q^2]_m}{[x^2q;q]_n[x^2y^2q^2;q^2]_m[q^2;q^2]_m} $$

$$ = \sum_{r=0}^{\infty}\frac{[x,-xq;q]_r(-zq)^r q^{-r(r+1)/2}}{[q,x^2q,xyq,-xyq;q]_r}\sum_{n=0}^{\infty}\frac{\Omega_{n+r}z^n q^{-nr}[q;q]_{n+r}[x^2y^2q;q]_{n+2r}}{[q;q]_n[x^2y^2q;q]_{n+r}} \qquad (3.1) $$

Proof of (3.1)

With the help of (2.6) the left side of (3.1) can be written as

$$ \sum_{n=0}^{\infty}\Omega_n z^n \sum_{r=0}^{n}\frac{[x;q]_r[-xq;q]_r q^r[q^{-n};q]_r[x^2y^2q^{1+n};q]_r}{[xyq;q]_r[-xyq;q]_r[q;q]_r[x^2q;q]_r} $$

Now, applying (2.14) it equals

$$ \sum_{r=0}^{\infty}\frac{[x,-xq;q]_r q^r z^r}{[q,x^2q,xyq,-xyq;q]_r}\sum_{n=0}^{\infty}\Omega_{n+r}z^n[q^{-n-r},x^2y^2q^{1+n+r};q]_r $$

$$= \sum_{r=0}^{\infty} \frac{[x, -xq; q]_r (-zq)^r q^{-r(r+1)/2}}{[q, x^2q, xyq, -xyq; q]_r} \sum_{n=0}^{\infty} \frac{\Omega_{n+r} z^n q^{-nr} [q; q]_{n+r} [x^2y^2q; q]_{n+2r}}{[q; q]_n [x^2y^2q; q]_{n+r}}$$

This prove (3.1).
Now, setting

$$\Omega_n = \frac{[x^2y^2q; q]_n [\alpha; q]_n}{[q; q]_n [\gamma; q]_n}$$

in (3.1), it equals, after some simplification,

$$\sum_{n=0}^{\infty} z^n \frac{[x^2y^2q; q]_n [\alpha; q]_n x^n [q; q]_n [x^2q^2; q^2]_m [y^2q^2; q^2]_m}{[q; q]_n [\gamma; q]_n [x^2q; q]_n [x^2y^2q^2; q^2]_m [q^2; q^2]_m}$$

$$= \sum_{r=0}^{\infty} \frac{[x, -xq, \alpha; q]_r [x^2y^2q; q]_{2r} (-zq)^r q^{-r(r+1)/2}}{[q, x^2q, xyq, -xyq, \gamma; q]_r} \, {}_2\Phi_1 \left[\begin{matrix} x^2y^2q^{1+2r}, \alpha q^r; q; z/q^r \\ \gamma q^2 \end{matrix} \right] \tag{3.2}$$

Now, setting $z = q^{1/2}/\alpha$ and $\gamma = x^2y^2q^2/\alpha$ in the above and summing the inner ${}_2\Phi_1$ with the help of (2.3), we get the following interesting transformations, after some simplification

$${}_4\Phi_3 \left[\begin{matrix} \alpha, \alpha q, y^2q^2, x^2y^2q; q^2; x^2q/\alpha^2 \\ x^2q, x^2y^2q^2/\alpha, x^2y^2q^3/\alpha \end{matrix} \right]$$

$$+ \frac{x\sqrt{q}(1-\alpha)(1-x^2y^2q)}{(1-x^2q)(\alpha - x^2y^2q^2)} \, {}_4\Phi_3 \left[\begin{matrix} \alpha q, \alpha q^2, y^2q^2, x^2y^2q^3; q^2; x^2q/\alpha^2 \\ x^2q^3, x^2y^2q^3/\alpha, x^2y^2q^4/\alpha \end{matrix} \right]$$

$$= \frac{[x^2y^2q, \sqrt{q}; q]_\infty [xyq/\alpha; \sqrt{q}]_\infty}{2[x^2y^2q^2/\alpha, \sqrt{q}/\alpha; q]_\infty [xy\sqrt{q}; \sqrt{q}]_\infty} \, {}_4\Phi_3 \left[\begin{matrix} x, -xq, \alpha, xy\sqrt{q}; q; q \\ x^2q, -xyq, \alpha\sqrt{q} \end{matrix} \right]$$

$$+ \frac{[x^2y^2q, \sqrt{q}; q]_\infty [-xyq/\alpha; \sqrt{q}]_\infty}{2[x^2y^2q^2/\alpha, \sqrt{q}/\alpha; q]_\infty [-xy\sqrt{q}; \sqrt{q}]_\infty} \, {}_4\Phi_3 \left[\begin{matrix} x, -xq, \alpha, -xy\sqrt{q}; q; q \\ x^2q, xyq, \alpha\sqrt{q} \end{matrix} \right] \tag{3.3}$$

Now, setting $z = q^{3/2}/\alpha$ and $\gamma = x^2y^2q^2/\alpha$ in (3.2) and summing the inner ${}_2\Phi_1$ with the help of (2.4), we are lead to the following interesting transformation involving four hypergeometric functions,

$${}_4\Phi_3 \left[\begin{matrix} \alpha, \alpha q, x^2y^2q, y^2q^2; q^2; q^3/\alpha^2 \\ x^2q, x^2y^2q^2/\alpha, x^2y^2q^3/\alpha \end{matrix} \right]$$

$$+ \frac{(1-\alpha)(1-x^2y^2q)xq^{3/2}}{(\alpha - x^2y^2q^2)(1-x^2q)} \, {}_4\Phi_3 \left[\begin{matrix} \alpha q, \alpha q^2, x^2y^2q^3, y^2q^2; q^2; q^3/\alpha^2 \\ x^2y^2q^3/\alpha, x^2y^2q^4/\alpha, x^2q^3 \end{matrix} \right]$$

$$= \frac{[x^2y^2q, \sqrt{q}; q]_\infty [xyq/\alpha; \sqrt{q}]_\infty}{2xy\sqrt{q}[x^2y^2q^2/\alpha, \sqrt{q}/\alpha; q]_\infty [xy\sqrt{q}; \sqrt{q}]_\infty} {}_4\Phi_3 \left[\begin{matrix} x, -xq, \alpha, xy\sqrt{q}; q; q \\ x^2q, -xyq, \alpha\sqrt{q} \end{matrix} \right]$$

$$- \frac{[x^2y^2q, \sqrt{q}; q]_\infty [-xyq/\alpha; \sqrt{q}]_\infty}{2xy\sqrt{q}[x^2y^2q^2/\alpha, \sqrt{q}/\alpha; q]_\infty [-xy\sqrt{q}; \sqrt{q}]_\infty} {}_4\Phi_3 \left[\begin{matrix} x, -xq, \alpha, -xy\sqrt{q}; q; q \\ x^2q, -xyq, \alpha\sqrt{q} \end{matrix} \right] \quad (3.4)$$

Again, if we put $z = q^{3/2}\alpha$ and $\gamma = x^2y^2q^3/\alpha$ in (3.2) and sum the inner ${}_2\Phi_1$ on the right with the help of (2.5), we get after some simplification,

$${}_4\Phi_3 \left[\begin{matrix} \alpha, \alpha q, x^2y^2q, y^2q^2; q^2; x^2q^3/\alpha^2 \\ x^2q, x^2y^2q^3/\alpha, x^2y^2q^4/\alpha \end{matrix} \right]$$

$$+ \frac{(1-\alpha)(1-x^2y^2q)xq^{3/2}/\alpha}{(1-x^2y^2q^3/\alpha)(1-x^2q)} {}_4\Phi_3 \left[\begin{matrix} \alpha q, \alpha q^2, x^2y^2q^3, y^2q^2; q^2; x^2q^3/\alpha^2 \\ x^2y^2q^4/\alpha, x^2y^2q^5/\alpha, x^2q^3 \end{matrix} \right]$$

$$= \frac{[x^2y^2q, \sqrt{q}, q^2/\alpha; q]_\infty [xyq^{3/2}; \sqrt{q}]_\infty}{2xy\sqrt{q}[x^2y^2q^3/\alpha, q/\alpha, q^{3/2}/\alpha; q]_\infty [xyq^{1/2}; \sqrt{q}]_\infty} {}_4\Phi_3 \left[\begin{matrix} x, -xq, xy\sqrt{q}, \alpha/q; q; q \\ x^2q, -xyq, \alpha/\sqrt{q} \end{matrix} \right]$$

$$- \frac{[x^2y^2q, \sqrt{q}, q^2/\alpha; q]_\infty [-xyq^{3/2}; \sqrt{q}]_\infty}{2xy\sqrt{q}[x^2y^2q^3/\alpha, q/\alpha, q^{3/2}/\alpha; q]_\infty [-xyq^{1/2}; \sqrt{q}]_\infty} \times$$

$${}_4\Phi_3 \left[\begin{matrix} x, -xq, -xy\sqrt{q}, \alpha/q; q; q \\ x^2q, -xyq, \alpha/\sqrt{q} \end{matrix} \right] \quad (3.5)$$

(ii) Next, using (2.7) in the following we get

$$\sum_{n=0}^{\infty} \Omega_n z^n \frac{x^{2n}[-q; q]_n [bq; q]_n}{[-x^2q; q]_n [bx^2q; q]_n}$$

$$= \sum_{n=0}^{\infty} \Omega_n z^n \sum_{r=0}^{n} \frac{[b^2x^4q^{2n+2}, q^{-2n}, x^2, x^2q; q^2; q^2]_r q^{2r}}{[q^2, bx^2q, bx^2q^2, x^4q^2; q^2]_r}$$

Now, using (2.14), the above relation takes the form

$$\sum_{n=0}^{\infty} \Omega_n z^n \frac{x^{2n}[-q; q]_n [bq; q]_n}{[-x^2q; q]_n [bx^2q; q]_n}$$

$$\sum_{r=0}^{\infty} \frac{[x^2, x^2q; q^2]_r (-zq^2)^r q^{-r(r+1)/2}}{[q^2, bx^2q, bx^2q^2, x^4q^2; q^2]_r} \sum_{n=0}^{\infty} \Omega_{n+r} z^n q^{-2nr} \frac{[q^2; q^2]_{n+r} [b^2x^4q^2; q^2]_{n+2r}}{[q^2; q^2]_n [b^2x^4q^2; q^2]_{n+r}} \quad (3.6)$$

If we set

$$\Omega_n = \frac{[b^2x^4q^2, \alpha; q^2]_n}{[q^2, \gamma; q^2]_n}$$

in (3.6), we get

$$\sum_{n=0}^{\infty} \frac{[b^2x^4q^2, \alpha; q^2]_n}{[q^2, \gamma; q^2]_n} z^n \frac{x^{2n}[-q; q]_n [bq; q]_n}{[-x^2q; q]_n [bx^2q; q]_n}$$

$$= \sum_{r=0}^{\infty} \frac{[x^2, x^2q, \alpha; q^2]_r (-zq^2)^r q^{-r(r+1)/2} [b^2x^4q^2; q^2]_{2r}}{[q^2, bx^2q, bx^2q^2, x^4q^2, \gamma; q^2]_r} \times$$

$${}_2\Phi_1 \left[\begin{array}{l} b^2x^4q^{4r+2}, \alpha q^{2r}; q^2; z/q^{2r} \\ \gamma q^{2r} \end{array} \right] \tag{3.7}$$

Now, taking $z = q/\alpha$ and $\gamma = b^2x^4q^4/\alpha$ in (3.7), we get after using (2.3)

$${}_4\Phi_3 \left[\begin{array}{l} bq, \sqrt{\alpha}, -\sqrt{\alpha}, -bx^2q; q; x^2q/\alpha \\ -x^2q, -bx^2q^2/\sqrt{\alpha}, bx^2q^2/\sqrt{\alpha} \end{array} \right]$$

$$= \frac{[q; q^2]_\infty [-bx^2q, bx^2q/\alpha; q]_\infty}{2[b^2x^4q^4/\alpha, q^2, q/\alpha; q^2]_\infty} \, {}_3\Phi_2 \left[\begin{array}{l} x^2, x^2q, \alpha; q^2; q^2 \\ x^4q^2, \alpha q \end{array} \right]$$

$$+ \frac{[q; q^2]_\infty [bx^2q, -bx^2q/\alpha; q]_\infty}{2[b^2x^4q^4/\alpha, q^2, q/\alpha; q^2]_\infty} \, {}_5\Phi_4 \left[\begin{array}{l} x^2, x^2q, -bx^2q, -bx^2q^2, \alpha; q^2; q^2 \\ x^4q^2, bx^2q, bx^2q^2, \alpha q \end{array} \right] \tag{3.8}$$

Further, if we set $z = q^3/\alpha$ and $\gamma = b^2x^4q^4/\alpha$ in (3.7) and use (2.4), we get the following transformation involving ${}_4\Phi_3$, ${}_3\Phi_2$ and ${}_5\Phi_4$

$${}_4\Phi_3 \left[\begin{array}{l} bq, \sqrt{\alpha}, -\sqrt{\alpha}, -bx^2q; q; x^2q^3/\alpha \\ -x^2q, bx^2q^2/\sqrt{\alpha}, -bx^2q^2/\sqrt{\alpha} \end{array} \right]$$

$$= \frac{[q; q^2]_\infty [-bx^2q, bx^2q/\alpha; q]_\infty}{2bx^2q[b^2x^4q^4/\alpha, q/\alpha; q^2]_\infty} \, {}_3\Phi_2 \left[\begin{array}{l} x^2, x^2q, \alpha; q^2; q^2 \\ x^4q^2, \alpha q \end{array} \right]$$

$$- \frac{[q; q^2]_\infty [bx^2q, -bx^2q/\alpha; q]_\infty}{2bx^2q[b^2x^4q^4/\alpha, q/\alpha; q^2]_\infty} \, {}_5\Phi_4 \left[\begin{array}{l} x^2, x^2q, -bx^2q, -bx^2q^2, \alpha; q^2; q^2 \\ x^4q^2, bx^2q, bx^2q^2, \alpha q \end{array} \right] \tag{3.9}$$

Again, putting $z = q^3/\alpha$ and $\gamma = b^2x^4q^6/\alpha$ in (3.7) and using (2.5), we get

$${}_4\Phi_3 \left[\begin{array}{l} bq, \sqrt{\alpha}, -\sqrt{\alpha}, -bx^2q; q; x^2q^3/\alpha \\ bx^2q^3/\alpha, -bx^2q^3/\alpha, -x^2q \end{array} \right]$$

$$= \frac{[-bx^2q, bx^2q^3/\alpha; q]_\infty [q, q^2/\alpha; q^2]_\infty}{2bx^2q[b^2x^4q^6/\alpha, q/\alpha, q^3/\alpha; q^2]_\infty} \, {}_4\Phi_3 \left[\begin{array}{l} x^2, x^2q, \alpha, \alpha; q^2; q \\ x^4q^2, \alpha q, \alpha/q \end{array} \right]$$

$$- \frac{[bx^2q, -bx^2q^3/\alpha; q]_\infty [q, q^2/\alpha; q^2]_\infty}{2bx^2q[b^2x^4q^6/\alpha, q/\alpha, q^3/\alpha; q^2]_\infty} \, {}_6\Phi_5 \left[\begin{array}{l} x^2, x^2q, \alpha, \alpha, -bx^2q, -bx^2q^2; q^2; q \\ bx^2q, bx^2q^2, x^4q^2, \alpha q, \alpha/q \end{array} \right] \tag{3.10}$$

(iii) Next, we have

$$\sum_{n=0}^{\infty}\Omega_n z^n \frac{x^n[q,bxq^2;q]_n[bx^2q^3,bq^2;q^2]_m[xq^2;q]_{2m}}{[xq,bx^2q^2;q]_n[q^2,x^2q^3;q^2]_m[bxq^2;q]_{2m}}$$

$$=\sum_{r=0}^{\infty}\frac{[x,-xq;q]_r(-zq)^r q^{-r(r+1)/2}}{[q,x^2q,xq\sqrt{b},-xq\sqrt{b};q]_r}\sum_{n=0}^{\infty}\Omega_{n+r}z^n q^{-nr}\frac{[q;q]_{n+r}[bx^2q^2;q]_{n+2r}}{[q;q]_n[bx^2q^2;q]_{n+r}} \qquad (3.11)$$

with help of (2.8) and (2.14).
Now taking

$$\Omega_n=\frac{[bx^2q^2,\alpha;q]_n}{[q,\gamma;q]_n}$$

in the above we get

$$\sum_{n=0}^{\infty}\frac{[bx^2q^2,\alpha;q]_n}{[q,\gamma;q]_n}\frac{x^n[q,bxq^2;q]_n[bx^2q^3,bq^2;q^2]_m[xq^2;q]_{2m}z^n}{[xq,bx^2q^2;q]_n[q^2,x^2q^3;q^2]_m[bxq^2;q]_{2m}}$$

$$=\sum_{r=0}^{\infty}\frac{[x,-xq,\alpha;q]_r[bx^2q^2;q]_{2r}(-zq)^r q^{-r(r+1)/2}}{[q,x^2q,xq\sqrt{b},-xq\sqrt{b},\gamma;q]_r}\,{}_2\Phi_1\left[\begin{matrix}bx^2q^{2+2r},\alpha q^r;z/q^r\\ \gamma q^r\end{matrix}\right] \qquad (3.12)$$

If we set $z=\sqrt{q}/\alpha$ and $\gamma=bx^2q^3/\alpha$ in (3.12) and use (2.3) to sum the inner ${}_2\Phi_1$ on the right, we get

$${}_5\Phi_4\left[\begin{matrix}\alpha,\alpha q,bx^2q^3,bq^2,xq^3;q^2;x^2q/\alpha^2\\ bx^2q^3/\alpha,bx^2q^4/\alpha,x^2q^3,xq\end{matrix}\right]$$

$$+\frac{(1-\alpha)(1-bxq^2)x\sqrt{q}/\alpha}{(1-xq)(1-bx^2q^3/\alpha)}\,{}_5\Phi_4\left[\begin{matrix}\alpha q,\alpha q^3,bxq^4,bx^2q^3,bq^2;q^2;x^2q/\alpha^2\\ bx^2q^4/\alpha,bx^2q^5/\alpha,x^2q^3,bxq^2\end{matrix}\right]$$

$$=\frac{[bx^2q^2,\sqrt{q};q]_\infty[\sqrt{b}xq^{3/2}/\alpha;\sqrt{q}]_\infty}{2[bx^2q^3/\alpha,\sqrt{q}/\alpha;q]_\infty[-\sqrt{b}xq;\sqrt{q}]_\infty}\,{}_4\Phi_3\left[\begin{matrix}x,-xq,\alpha,xq^{3/2}\sqrt{b};q;q\\ x^2q^2,-xq\sqrt{b},\alpha\sqrt{q}\end{matrix}\right]$$

$$+\frac{[bx^2q^2,\sqrt{q};q]_\infty[-\sqrt{b}xq^{3/2}/\alpha;\sqrt{q}]_\infty}{2[bx^2q^3/\alpha,\sqrt{q}/\alpha;q]_\infty[-\sqrt{b}xq;\sqrt{q}]_\infty}\,{}_4\Phi_3\left[\begin{matrix}x,-xq,\alpha,xq^{3/2}\sqrt{b};q;q\\ x^2q^2,xq\sqrt{b},\alpha\sqrt{q}\end{matrix}\right] \qquad (3.13)$$

Further, putting $z=q^{3/2}/\alpha$ and $\gamma=bx^2q^3/\alpha$ in (3.12) and using (2.4), we get

$${}_5\Phi_4\left[\begin{matrix}\alpha,\alpha q,bx^2q^3,bq^2,xq^3;q^2;x^2q^3/\alpha^2\\ bx^2q^3/\alpha,bx^2q^4/\alpha,x^2q^3,xq\end{matrix}\right]$$

$$+\frac{(1-\alpha)(1-bxq^2)xq^{3/2}/\alpha}{(1-xq)(1-bx^2q^3/\alpha)}\,{}_5\Phi_4\left[\begin{array}{l}\alpha q,\alpha q^3,bxq^4,bx^2q^3,bq^2;q^2;x^2q^3/\alpha^2\\ bx^2q^4/\alpha,bx^2q^5/\alpha,x^2q^3,bxq^2\end{array}\right]$$

$$=\frac{[bx^2q^2,\sqrt{q};q]_\infty[\sqrt{b}xq^{3/2};\sqrt{q}]_\infty}{2\sqrt{b}xq[bx^2q^3/\alpha,\sqrt{q}/\alpha;q]_\infty[\sqrt{b}xq;\sqrt{q}]_\infty}\,{}_4\Phi_3\left[\begin{array}{l}x,-xq,\alpha,-xq^{3/2}\sqrt{b};q;q\\ x^2q^2,-xq\sqrt{b},\alpha\sqrt{q}\end{array}\right]$$

$$-\frac{[bx^2q^2,\sqrt{q};q]_\infty[-\sqrt{b}xq^{3/2};\sqrt{q}]_\infty}{2\sqrt{b}xq[bx^2q^3/\alpha,\sqrt{q}/\alpha;q]_\infty[-\sqrt{b}xq;\sqrt{q}]_\infty}\,{}_4\Phi_3\left[\begin{array}{l}x,-xq,\alpha,xq^{3/2}\sqrt{b};q;q\\ x^2q^2,xq\sqrt{b},\alpha\sqrt{q}\end{array}\right] \quad (3.14)$$

Again, taking $z=q^{3/2}/\alpha$ and $\gamma=bx^2q^4/\alpha$ in (2.13) and using (2.5), we get

$$\sum_{n=0}^{\infty}\frac{[\alpha,bxq^2;q]_n[bx^2q^3,bq^2;q^2]_m[xq^2;q]_{2m}(xq^{3/2}/\alpha)^n}{[bx^2q^4/\alpha,xq;q]_n[q^2,x^2q^3;q^2]_m[bxq^2;q]_{2m}}$$

$$=\frac{[bx^2q^2,\sqrt{q},q^2/\alpha;q]_\infty[\sqrt{b}xq^2;\sqrt{q}]_\infty}{2xq\sqrt{b}[bx^2q^4/\alpha,q/\alpha,q^3/\alpha;q]_\infty[\sqrt{bxq};\sqrt{q}]_\infty}\times$$

$${}_5\Phi_4\left[\begin{array}{l}x,-xq,\alpha/q,\sqrt{b}xq,\sqrt{b}xq^{3/2};q;q\\ x^2q,xq\sqrt{b},-xq\sqrt{b},\alpha/\sqrt{q}\end{array}\right]$$

$$-\frac{[bx^2q^2,\sqrt{q},q^2/\alpha;q]_\infty[-\sqrt{b}xq^2/\alpha;\sqrt{q}]_\infty}{2xy\sqrt{b}[bx^2q^4/\alpha,q/\alpha,q^{3/2}/\alpha;q]_\infty[-xq\sqrt{b};\sqrt{b}]_\infty}\times$$

$${}_5\Phi_4\left[\begin{array}{l}x,-xq,\alpha/q,-\sqrt{b}xq,-\sqrt{b}xq^{3/2};q;q\\ x^2q,xq\sqrt{b},-xq\sqrt{b},\alpha/\sqrt{q}\end{array}\right] \quad (3.15)$$

(iv) Now, using (2.9) and (2.14), we can prove that

$$\sum_{n=0}^{\infty}\Omega_n z^n\frac{x^{n-m}[aq/x,q;q]_n[aq^2,xq;q^2]_m}{q^m[aq,xq;q]_n[q^2,aq/x;q^2]_m}$$

$$=\sum_{r=0}^{\infty}\frac{[x,\sqrt{x/q},-\sqrt{x/q},q^{-n},aq^{1+n};q]_rq^r}{[q,\sqrt{aq},-\sqrt{aq},x/q,xq;q]_r}\sum_{n=0}^{\infty}\frac{[q;q]_{n+r}[aq;q]_{n+2r}q^{-nr}}{[q;q]_n[aq;q]_{n+r}}$$

which with

$$\Omega_n=\frac{[aq,\alpha;q]_n}{[q,\gamma;q]_n}$$

takes the form

$$\sum_{n=0}^{\infty}\frac{[aq,\alpha;q]_n}{[q,\gamma;q]_n}\frac{x^{n-m}[aq/x,q;q]_n[aq^2,xq;q^2]_mz^n}{q^m[aq,xq;q]_n[q^2,aq/x;q^2]_m}$$

$$= \sum_{r=0}^{\infty} \frac{[x, \sqrt{x/q}, -\sqrt{x/q}, \alpha; q]_r [aq; q]_{2r} (-zq)^r q^{-r(r+1)/2}}{[q, \sqrt{aq}, -\sqrt{aq}, x/q, xq, \gamma; q]_r} \times$$

$$ {}_2\Phi_1 \left[\begin{array}{l} aq^{1+r}, \alpha q^r; q; z/q^r \\ \gamma q^r \end{array} \right] \tag{3.16}$$

If we set $z = q^{3/2}/\alpha$ and $\gamma = aq^2/\alpha$ in (3.16) and use (2.4), we get

$$ {}_4\Phi_3 \left[\begin{array}{l} \alpha, \alpha q, aq^2/x, aq^2; q^2; xq^2/\alpha^2 \\ aq^2/\alpha, aq^3/\alpha, xq^2 \end{array} \right]$$

$$+ \frac{(1-\alpha)(1-aq/x)xq^{3/2}/\alpha}{(1-xq)(1-aq^2/\alpha)} \, {}_6\Phi_5 \left[\begin{array}{l} \alpha q, \alpha q^2, \alpha q^2/x, \alpha q^3/x, aq^2, xq; xq^2/\alpha^2 \\ aq^3/\alpha, aq^4/\alpha, xq^2, xq^3, aq/x \end{array} \right]$$

$$= \frac{[aq, \sqrt{q}; q]_\infty [q\sqrt{a}/\alpha; \sqrt{q}]_\infty}{2\sqrt{aq}[aq^2/\alpha, \sqrt{q}/\alpha; q]_\infty [\sqrt{aq}; \sqrt{q}]_\infty} \times$$

$$ {}_5\Phi_4 \left[\begin{array}{l} x, \sqrt{x/q}, -\sqrt{x/q}, \alpha, q\sqrt{a}; q; q \\ -\sqrt{xa}, x/q, xq, \alpha\sqrt{q} \end{array} \right]$$

$$- \frac{[aq, \sqrt{q}; q]_\infty [-q\sqrt{a}/\alpha; \sqrt{q}]_\infty}{2\sqrt{aq}[aq^2/\alpha, \sqrt{q}/\alpha; q]_\infty [-\sqrt{aq}; \sqrt{q}]_\infty} \times$$

$$ {}_5\Phi_4 \left[\begin{array}{l} x, \sqrt{x/q}, -\sqrt{x/q}, \alpha, -q\sqrt{a}; q; q \\ \sqrt{xa}, x/q, xq, \alpha/\sqrt{q} \end{array} \right] \tag{3.17}$$

Again, using (2.10) and (2.14), we easily get

$$\sum_{n=0}^{\infty} \Omega_n z^n \frac{a^n [q^3; q^3]_n [aq; q]_n}{[a^3q^3; q^3]_n [q; q]_n}$$

$$= \sum_{r=0}^{\infty} \frac{[a, aq, aq^2; q^3]_r (-zq^3)^r q^{-3r(r+1)/2}}{[q^3, (aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^2, -a^{3/2}q^2; q^3]_r} \times$$

$$\sum_{n=0}^{\infty} \Omega_{n+r} z^n \frac{[a^3q^3; q^3]_{n+2r} [q^3; q^3]_{n+r} q^{-3nr}}{[a^3q^3; q^3]_{n+r} [q^3; q^3]_n}$$

Now, setting

$$\Omega_n = \frac{[a^3q^3, \alpha; q^3]_n}{[q^3, \gamma; q^3]_n}$$

in the above and taking $z = q^{9/2}/\alpha$ and $\gamma = a^3q^6/\alpha$ in it we get the following transformation with the help of (2.4),

$$
{}_4\Phi_3\left[\begin{array}{l}\alpha^{1/3}, \omega\alpha^{1/3}, \omega^2\alpha^{1/3}, aq; q; aq^{9/2}/\alpha \\ aq^2/\alpha^{1/3}, \omega aq^2/\alpha^{1/3}, \omega^2 aq^2/\alpha^{1/3}\end{array}\right]
$$

$$
= \frac{[q^{3/2}, a^3q^3; q^3]_\infty [a^{3/2}q^3\alpha; q^{3/2}]_\infty}{a(aq)^{3/2}[a^3q^6/\alpha, q^{3/2}/\alpha; q^3]_\infty [(aq)^{3/2}; q^{3/2}]_\infty} \times
$$

$$
{}_5\Phi_4\left[\begin{array}{l}q, aq, aq^2, \alpha, a^{3/2}q^3; q^3; q^3 \\ -(aq)^{3/2}, a^{3/2}q^2, -a^{3/2}q^2, \alpha q^{3/2}\end{array}\right]
$$

$$
- \frac{[q^{3/2}, a^3q^3; q^3]_\infty [-a^{3/2}q^3\alpha; q^{3/2}]_\infty}{a(aq)^{3/2}[a^3q^6/\alpha, q^{3/2}/\alpha; q^3]_\infty [-(aq)^{3/2}; q^{3/2}]_\infty} \times
$$

$$
{}_5\Phi_4\left[\begin{array}{l}\alpha^{1/3}, \omega\alpha^{1/3}, \omega^2\alpha^{1/3}, \alpha, -a^{3/2}q^3; q^3; q^3 \\ (aq)^{3/2}, a^{3/2}q^2, -a^{3/2}q^2, \alpha q^{3/2}\end{array}\right] \tag{3.18}
$$

Further, using (2.13) and (2.14), we easily get

$$
\sum_{n=0}^{\infty} \Omega_n z^n \frac{[q, \sqrt{a}; q]_n [aq^3, q^6\sqrt{a}; q^3]_m a^{(n-m)/2}}{[aq, q^2\sqrt{a}; q]_n [q^3, \sqrt{a}; q^3]_m}
$$

$$
= \sum_{r=0}^{\infty} \frac{[a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}; q]_r (-zq)^r q^{-r(r+1)/2}}{[q, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a}; q]_r} \sum_{n=0}^{\infty} \Omega_{n+r} z^n q^{-nr} \frac{[q; q]_{m+r}[aq; q]_{n+2r}}{[q; q]_n [aq; q]_{n+r}}
$$

Now, setting

$$
\Omega_n = \frac{[aq, \alpha; q]_n}{[q, \gamma; q]_n}
$$

in the above and taking $z = q^{1/2}/\alpha$ and $\gamma = aq^2/\alpha$ in it we get the following transformation with the help of (2.3),

$$
{}_6\Phi_5\left[\begin{array}{l}\alpha, \alpha q, \alpha q^2, aq^3, q^6\sqrt{a}, q\sqrt{a}; q^3; aq^{3/2}/\alpha^3 \\ aq^2/\alpha, aq^3/\alpha, aq^4/\alpha, q^3/\sqrt{a}, q^4\sqrt{a}\end{array}\right]
$$

$$
+ \frac{(1-\alpha)(1-\sqrt{a})\sqrt{aq}/\alpha}{(1-q^2\sqrt{a})(1-aq^2/\alpha)}\, {}_7\Phi_6\left[\begin{array}{l}\alpha q, \alpha q^2, \alpha q^3, aq^3, q^6\sqrt{a}, q\sqrt{a}, q^2\sqrt{a}; q^3; aq^{3/2}/\alpha^3 \\ aq^3/\alpha, aq^4/\alpha, aq^5/\alpha, \sqrt{a}, q^4/\sqrt{a}, q^5\sqrt{a}\end{array}\right]
$$

$$
+ \frac{(1-\alpha)(1-\sqrt{a})(1-q\sqrt{a})aq/\alpha^2}{(1-aq^2/\alpha)(1-aq^3/\alpha)(1-q^2\sqrt{a})(1-q^3\sqrt{a})} \times
$$

$$
{}_6\Phi_5\left[\begin{array}{l}\alpha q^2,\alpha q^3,\alpha q^4,q^2\sqrt{a},q^3\sqrt{a},aq^3;q;aq^{3/2}/\alpha^3\\ aq^4/\alpha,aq^5/\alpha,aq^6/\alpha,q^5\sqrt{a},\sqrt{a}\end{array}\right]
$$

$$
=\frac{[\sqrt{q},aq;q]_\infty[q\sqrt{a}/\alpha;\sqrt{q}]_\infty}{2[aq^2/\alpha,\sqrt{q}/\alpha;q]_\infty[\sqrt{aq};\sqrt{q}]_\infty}\times
$$

$$
{}_6\Phi_5\left[\begin{array}{l}a^{1/3},\omega a^{1/3},\omega^2a^{1/3},\alpha,q\sqrt{a},-q\sqrt{a};q;q\\ \sqrt{a},-\sqrt{a},-q\sqrt{a},q^2\sqrt{a},\alpha\sqrt{q}\end{array}\right]
$$

$$
+\frac{[\sqrt{q},aq;q]_\infty[-q\sqrt{a}/\alpha;\sqrt{q}]_\infty}{2[aq^2/\alpha,\sqrt{q}/\alpha;q]_\infty[-\sqrt{aq};\sqrt{q}]_\infty}\times
$$

$$
{}_6\Phi_5\left[\begin{array}{l}a^{1/3},\omega a^{1/3},\omega^2a^{1/3},\alpha,q\sqrt{a},-q\sqrt{a};q;q\\ \sqrt{a},-\sqrt{a},-q\sqrt{a},q^2\sqrt{a},\alpha\sqrt{q}\end{array}\right] \tag{3.19}
$$

It is evident that, using the present technique, a variety of interesting transformations can be established.

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