South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752**
Vol.11, No.2 2015, pp. 59-68

Common fixed point results for weakly commuting maps by altering distances

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Abstract: The present paper deals with common fixed point results for four mappings and also for a sequence of mappings on a metric space under the control function namely the altering distances between points. The results obtained generalize the earlier results of Khan, Swaleh and Sessa (1984) and Sastry and Babu (1999, 2001) and others in turn.
Keywords: Weakly commuting map, altering distance, fixed point.
AMS Subject classification: 47H10, 54H25.

1. Introduction

The theory of common fixed point for self mappings in metric space satisfying certain conditions has a vast literature. However the existence of fixed points for self maps on a metric space by altering distances between the points with the use of certain control function is an interesting aspect. In this direction Khan, Swaleh and Sessa [2] established the existence and uniqueness of a fixed point for a single altering distance map. Recently Sastry and Babu [7,8], Naidu [5,6] proved a fixed point theorem by altering distances between the points for a pair of self maps, which address a new type of contractive fixed point problems. Pant [3] established a unique common fixed point theorem for four self maps by using the conditions of the type commutativity, contractive and continuity. The main purpose of this paper is to obtain conditions for the existence of a unique common fixed point for four self maps on a complete metric space by altering distances between the points. Before going to our results, we give here some definitions.

Definition 1.1 [2] An altering distance is a mapping $\varphi : [0,\infty) \rightarrow [0,\infty)$ which satisfies

1. φ is increasing and continuous,
2. $\varphi(t) = 0$ if and only if $t = 0$.

Definition 1.2 Let A and S be self mappings of a metric space (X, d), then A and

S are said to be reciprocally continuous at a point t in X if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$.

Definition 1.3 Two self maps A and S of a metric space (X,d) are called Ψ-compatible if $\lim_{n\to\infty} \Psi(d(ASx_n, SAx_n)) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

Definition 1.4 [1] Two self maps f and g on a metric space (X,d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

2. Main Results

Theorem 2.1

Let $\{A, S\}$ and $\{B, T\}$ be weakly commuting pairs of self maps of a complete metric space (X, d) and $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing function, satisfying $\Psi(2t) \leq 2\Psi(t)$ and $\Psi(t) = 0$ if and only if $t = 0$, such that

$$AX \subset TX, BX \subset SX \tag{1}$$

$$\Psi(d(Ax, By)) \leq hM\Psi(x, y) \quad \forall x, y \in X \tag{2}$$

where $M\Psi(x, y) = max\{\Psi(d(Sx, Ty)), \Psi(d(Ax, Sx)), \Psi(d(By, Ty)), \Psi(d(Ax, Ty)), \Psi(d(Ax, By))\}$, $0 \leq h < 1$. If $\{A, S\}$ or $\{B, T\}$ is a Ψ-compatible pair of reciprocally continuous mappings, then A,B,S and T have a unique common fixed point.

Proof

Let x_0 be any point in X. Let $\{x_n\}$ be a sequence in X. Then by (1) we can define another sequence $\{y_n\}$ such that for $n = 0, 1, 2, ...$

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \tag{3}$$

We now show that $\{y_n\}$ is a Cauchy sequence.
From (2), we have

$$\Psi(d(y_{2n}, y_{2n+1})) = \Psi(d(Ax_{2n}, Bx_{2n+1}))$$

$$\leq hM\Psi(x_{2n}, x_{2n+1})$$

$$= h\ max\,\{\Psi(d(Sx_{2n}, Tx_{2n+1})), \Psi(d(Ax_{2n}, Sx_{2n})), \Psi(d(Bx_{2n+1}, Tx_{2n+1})),$$

$$\Psi(d(Ax_{2n}, Tx_{2n+1})), \Psi(d(Ax_{2n}, Bx_{2n+1}))\}$$

$$= h\ max\,\{\Psi(d(y_{2n-1}, y_{2n})), \Psi(d(y_{2n}, y_{2n-1})), \Psi(d(y_{2n+1}, y_{2n})),$$

$$\Psi(d(y_{2n}, y_{2n})), \Psi(d(y_{2n+1}, y_{2n}))\}$$

Now, if max $\{d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n})\} = d(y_{2n+1}, y_{2n})$ then

$$\Psi(d(y_{2n}, y_{2n+1})) \leq h\Psi(d(y_{2n}, y_{2n+1})),$$

i.e., $(1-h)\Psi(d(y_{2n}, y_{2n+1})) \leq 0$,
i.e., $\Psi(d(y_{2n}, y_{2n+1})) = 0$, which is a contradiction.
So we must have max $\{d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n})\} = d(y_{2n-1}, y_{2n})$.
Therefore,

$$\Psi d(y_{2n}, y_{2n+1}) \leq h\Psi(d(y_{2n-1}, y_{2n})) \tag{4}$$

In a similar way, we can also show that

$$\Psi(d(y_{2n-1}, y_{2n})) \leq h\Psi(d(y_{2n-2}, y_{2n-1})) \tag{5}$$

By repeated application of (4) and (5), we get $\Psi(d(y_n, y_{n+1})) \leq h^n\Psi(d(y_0, y_1))$. Moreover for every positive integer p, we have

$$\begin{aligned}\Psi(d(y_n, y_{n+p})) &\leq \Psi[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n+p-1}, y_{n+p})] \\ &\leq \Psi[(1 + h + ... + h^{p-1})h^n d(y_0, y_1)] \\ &\leq \Psi\left[\left(\frac{1}{1-h}\right) h^n d(y_0, y_1)\right].\end{aligned}$$

Now for a given $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$\Psi\left[\left(\frac{1}{1-h}\right) h^n d(y_0, y_1)\right] < \Psi(\varepsilon), \qquad \forall n \geq N.$$

This implies that $d(y_n, y_{n+p}) < \varepsilon$ for all $n \geq N$.
Hence, $\{y_n\}$ is a Cauchy sequence in X. Since X complete, there is a point z in X such that $y_n \to z$ as $n \to \infty$.
Hence from (3), we have

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z. \tag{6}$$

Now, suppose that $\{A, S\}$ is a $\Psi-$ compatible pair of reciprocally continuous mappings.
Since A and S are reciprocally continuous mappings, by (6), we get

$$ASx_{2n} \to Az, \qquad SAx_{2n} \to Sz. \tag{7}$$

$\Psi-$ compatibility of A and S implies that $\lim_{n\to\infty} \Psi(d(ASx_{2n}, SAx_{2n})) = 0$.
We now show that $Az = Sz$.
Suppose $Az \neq Sz$.
Let $d(Az, Sz) = 2\varepsilon$, then there exists $N \in \mathbb{Z}^+$ such that $\Psi(d(ASx_{2n}, SAx_{2n})) < \Psi(\varepsilon)$ for all $n \geq N$.

This implies that $d(ASx_{2n}, SAx_{2n}) < \varepsilon$ for all $n \geq N$. Hence by (6), $d(Az, Sz) < \varepsilon = \frac{1}{2}d(Az, Sz)$, a contradiction. Hence

$$Az = Sz. \tag{8}$$

Since $AX \subset TX$, there is a point w in X such that $Tw = Az$. By (8) we have

$$Tw = Az = Sz. \tag{9}$$

We now show that $Az = Bw$. Suppose $Az \neq Bw$. By (2) we have $\Psi(d(Az, Bw)) \leq hM\Psi(z, w) = h\Psi d(Bw, Az)$, a contradiction.
Hence $Az = Bw$. Therefore by (9), we get

$$Bw = Az = Sz = Tw. \tag{10}$$

Since A and S are weakly commuting, we have by (10)

$$ASz = SAz, \quad AAz = ASz = SAz = SSz. \tag{11}$$

Since B and T are weakly commuting, we have

$$BBw = BTw = TBw = TTw. \tag{12}$$

We now show that $AAz = Az$. Suppose $AAz \neq Az$, then by (2), we have
$\Psi(d(Az, AAz)) = \Psi(d(Bw, AAz))$
$\leq hM\Psi(Az, w)$
$= h\Psi(d(Az, AAz))$, by (9) & (10), a contradiction.
Hence, AAz=Az
Also $AAz = SAz$.
Therefore Az is a common fixed point of A and S.
Suppose that $BBw \neq Bw$, then by (2), (10) and (12) we have
$\Psi(d(Bw, BBw)) = \Psi(d(Az, BBw))$
$\leq hM\Psi(z, Bw)$
$= h\Psi(d(Bw, BBw))$
$< \Psi(d(Bw, BBw))$, a contradiction.

Hence, $BBw = Bw$ and since $TBw = BBw$, we have Bw is a common fixed point for B and T.
Since $Az = Bw$, we have Az is a common fixed point for A,B,S and T.
Uniqueness of a common fixed point follows from (2).

The proof is similar when the pair $\{B, T\}$ is assumed to be Ψ-compatible and reciprocally continuous.

For a sequence of mappings we have the following common fixed point result.

Theorem 2.2

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of self maps on (X, d). Assume that

(A1): There exists a ϕ in Φ, where Φ denotes the set of all continuous self maps ϕ of $\mathbb{R}^+$ satisfying, ϕ is increasing and $\phi(t) = 0$ iff $t = 0$, such that

$$\phi(d(T_i x, T_j y)) \leq a\phi(d(x, y)) + b(\phi(d(x, T_i x)) + \phi(d(y, T_j y)))$$

$$+c(\phi(d(x, T_j y)) + \phi(d(y, T_i x)))$$

for all $i, j \in \mathbb{N}$ and for all distinct $x, y \in X$, where $a, c \geq 0$, $0 < b < 1$ with $a + 2b + 2c < 1$.

(A2): There is a point x_0 in X such that any two consecutive members of the sequence $\{x_n\}$ defined by $x_n = T_n x_{n-1}$, $n \geq 1$ are distinct.

Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X. Infact, $\{x_n\}$ is Cauchy sequence and the limit of $\{x_n\}$ is the unique common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Proof

Let $\alpha_n = d(x_n, x_{n+1})$ and $\beta_n = \phi(\alpha_n)$. From $(A1)$ and $(A2)$, we have
$\beta_1 = \phi(d(x_1, x_2))$
$= \phi(d(T_1 x_0, T_2 x_1))$

$$\leq a\phi(d(x_0, x_1)) + b(\phi(d(x_0, x_1)) + \phi(d(x_1, x_2)))$$

$$+c(\phi(d(x_0, x_2)) + \phi(d(x_1, x_1)))$$

or,

$$\phi(d(x_1, x_2)) \leq a\phi(d(x_0, x_1)) + b(\phi(d(x_0, x_1)) + \phi(d(x_1, x_2)))$$

$$+c(\phi(d(x_0, x_1)) + \phi(d(x_1, x_2)))$$

or, $\beta_1 \leq a\beta_0 + b\beta_0 + b\beta_1 + c\beta_0 + c\beta_1$
or, $\beta_1 \leq \dfrac{(a+b+c)}{(1-b-c)}\beta_0 = k\beta_0$, where $k = \dfrac{(a+b+c)}{(1-b-c)} < 1$ since $a + 2b + 2c < 1$.
By induction, it follows that

$$\beta_n \leq k\beta_{n-1}, \quad \forall n \geq 1. \tag{13}$$

As such $\beta_n \downarrow 0$ as $n \to \infty$ and since $\alpha_n < \alpha_{n-1}$ for $n = 1, 2, ...$ and therefore, $\{\alpha_n\}$ is a decreasing sequence of non-negative real numbers.
Thus

$$\{\alpha_n\} \downarrow 0. \tag{14}$$

The remaining part of the theorem is to show that $\{x_n\}$ is Cauchy sequence in X. If not so, then there is an $\varepsilon > 0$ and sequence $\{m(k)\}$ and $\{n(k)\}$ such that

$$m(k) < n(k), \quad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \quad \text{and} \quad d(x_{n(k)-1}, x_{m(k)}) < \varepsilon.$$

Assume that $x_{n(k)-1} = x_{m(k)-1}$ for infinitely many k. Then for such k, we have

$$\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

$$= d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \to 0 \text{ as } \; k \to \infty, \quad \text{a contradiction.}$$

Hence, for large $k, x_{n(k)-1} \neq x_{m(k)-1}$.
Consequently,

$$\phi(\varepsilon) \leq \phi(d(x_{n(k)}, x_{m(k)})) = \phi(d(T_{n(k)}x_{n(k)-1}, T_{m(k)}x_{m(k)-1}))$$

$$\leq a\phi(d(x_{n(k)-1}, x_{m(k)-1})) + b(\phi(d(x_{n(k)-1}, x_{n(k)})) + \phi(d(x_{m(k)-1}, x_{m(k)})))$$
$$+c(\phi(d(x_{n(k)-1}, x_{m(k)})) + \phi(d(x_{m(k)-1}, x_{n(k)})))$$

$\leq a\phi(d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}))$
$+b(\phi(d(x_{n(k)-1}, x_{n(k)})) + \phi(d(x_{m(k)-1}, x_{m(k)})))$
$+c(\phi(d(x_{n(k)-1}, x_{m(k)})) + \phi(d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})))$
$\leq a\phi(\varepsilon + d(x_{m(k)}, x_{m(k)-1}))$
$+b(\phi(d(x_{n(k)-1}, x_{n(k)})) + \phi(d(x_{m(k)-1}, x_{m(k)})))$
$+c(\phi(\varepsilon) + \phi(d(x_{m(k)-1}, x_{m(k)})) + \phi(\varepsilon) + \phi(d(x_{n(k)-1}, x_{n(k)})))$
$\to a\phi(\varepsilon) + 2c\phi(\varepsilon)$ as $k \to \infty$ by equation (14).
Hence, $\phi(\varepsilon) \leq (a + 2c)\phi(\varepsilon) < \phi(\varepsilon)$, a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence in X.

As X is complete, limit of $\{x_n\}$ exists and there is a sequence $\{n(k)\}$ such that $y \neq x_{n(k)-1}$ otherwise, $y = x_{n-1}$ for large n, which is not the case, since consecutive terms are different.

With this subsequence $\{x_{n(k)}\}$, we have for any positive integer m,

$$\phi(d(T_m y, x_{n(k)})) = \phi(d(T_m y, T_{n(k)}x_{n(k)-1}))$$

$$\leq a\phi(d(y, x_{n(k)-1})) + b(\phi(d(T_m y, y)) + \phi(d(x_{n(k)}, x_{n(k)-1})))$$

$$+c(\phi(d(T_m y, x_{n(k)-1})) + \phi(d(x_{n(k)}, y))).$$

Taking limits as $k \to \infty$, we have

$$\phi(d(T_m y, y)) \leq b\phi(d(T_m y, y)) + c\phi(d(T_m y, y)) = (b+c)\phi(d(T_m y, y)).$$

Since $0 < b < 1$ and $0 < c < 1$, it follows that $\phi(d(T_m y, y)) = 0$ and so $d(T_m y, y) = 0$.

Thus y is a common fixed point for the sequence $\{T_n\}_{n=1}^{\infty}$

Uniqueness of the fixed point follows trivially from $(A1)$.

The nest result deals with a common fixed point theorem for a sequence of self mappings satisfying a more general inequality condition

Theorem 2.3

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of self maps on (X, d) and assume there is a point x_0 in X such that any two consecutive members of the sequence $\{x_n\}$ defined by $x_n = T_n x_{n-1},\ n \geq 1$ are distinct. Further, assume that

(A3): There exists ϕ in Φ where Φ denotes the set of all continuous self maps of $\mathbb{R}^+$ satisfying

1. ϕ is increasing
2. $\phi(t) = 0$ iff $t = 0$
3. $\phi(x+y) \leq \phi(x) + \phi(y)$ such that

$$\phi(d(T_i x, T_j y)) \leq a \max\{\phi(d(x,y)), \phi(d(x, T_i x)), \phi(d(y, T_j y)), \phi(d(x, T_j y)),$$
$$\phi(d(y, T_i x))\} + b(\phi(d(x, T_i x)) + \phi(d(y, T_j y)))$$

for some $0 < a < 1, b > 0$ with $a + b < 1/2$ and for all $i, j \in \mathbb{N}$ and all distinct $x, y \in X$. Then the sequence $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X. In fact $\{x_n\}$ is Cauchy sequence and the limit of $\{x_n\}$ is the unique common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Proof:

Let $\alpha_n = d(x_n, x_{n+1})$ and $\beta_n = \phi(\alpha_n)$. We have form (A3)

$\beta_1 = \phi(d(x_1, x_2))$

$= \phi(d(T_1 x_0, T_2 x_1))$

$\leq a \max\{\phi(d(x_0, x_1)), \phi(d(x_0, x_1)), \phi(d(x_1, x_2)), \phi(d(x_0, x_2)), \phi(d(x_1, x_1))\}$

$+b(\phi(d(x_0, x_1)) + \phi(d(x_1, x_2)))$

$\leq a \max\{\phi(d(x_0, x_1)), \phi(d(x_0, x_1)), \phi(d(x_1, x_2)),$

$\phi(d(x_0, x_1)) + (d(x_1, x_2)), \phi(d(x_1, x_1))\} + b(\phi(d(x_0, x_1)) + \phi(d(x_1, x_2)))$

$\leq a \max\{\phi(\alpha_0), \phi(\alpha_0), \phi(\alpha_1), \phi(\alpha_0 + \alpha_1), \phi(0)\} + b\beta_0 + b\beta_1$

$\leq a(\beta_0+\beta_1)+b\beta_0+b\beta_1 \leq (a+b)\beta_0+(a+b)\beta_1$ or, $\beta_1 \leq \left(\dfrac{a+b}{1-a-b}\right)\beta_0 = k\beta_0$,
where $k = \left(\dfrac{a+b}{1-a-b}\right) < 1$.
By induction, it follows that

$$\beta_n \leq k\beta_{n-1}, \quad \forall n \geq 1, \quad \beta_n = k^n\beta_0. \tag{15}$$

As such $\beta_n \downarrow 0$ as $n \to \infty$ and $\alpha_n < \alpha_{n-1}$, for $n = 1, 2, ...$

Therefore $\{\alpha_n\}$ is a decreasing sequence of non-negative real numbers. Thus $\{\alpha_n\} \downarrow \alpha$ (say) and so $\beta_n = \phi(\alpha_n) \downarrow \phi(\alpha)$. Consequently $\phi(\alpha) = 0$ and hence $\alpha = 0$.
Therefore,

$$\{\alpha_n\} \downarrow 0. \tag{16}$$

We now show that the sequence $\{x_n\}$ is Cauchy.

If not so, then there exists a $\varepsilon > 0$ and sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $k \leq n(k) < m(k)$ such that

$$d_k = d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \tag{17}$$

Let $m(k)$ be the least integer exceeding $n(k)$ for which (17) is true, then by well ordering principle we have $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$.

Now,

$$\varepsilon < d_k \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) < d(x_{m(k)}, x_{m(k)-1}) + \varepsilon \to \varepsilon$$

as $k \to \infty$ and $d_k \to \varepsilon$. Hence, for large $k, x_{m(k)-1} \neq x_{n(k)-1}$.
Consequently

$$\phi(\varepsilon) \leq \phi(d(x_{m(k)}, x_{n(k)})) = \phi(d(T_{m(k)}x_{m(k)-1}, T_{n(k)}x_{n(k)-1}))$$

$$\leq a \max\{\phi(d(x_{m(k)-1}, x_{n(k)-1})), \phi(d(x_{m(k)-1}, x_{m(k)})), \phi(d(x_{n(k)-1}, x_{n(k)})),$$
$$\phi(d(x_{m(k)-1}, x_{n(k)})), \phi(d(x_{n(k)-1}, x_{m(k)}))\}$$
$$+b(\phi(d(x_{m(k)-1}, x_{m(k)})) + \phi(d(x_{n(k)-1}, x_{n(k)})))$$
$$\leq a\phi(\varepsilon + d(x_{n(k)}, x_{n(k)-1})) + b(\phi(d(x_{m(k)-1}, x_{m(k)})) + \phi(d(x_{n(k)-1}, x_{n(k)}))) \to a\phi(\varepsilon)$$

as $k \to \infty$., by (16).

Hence $\phi(\varepsilon) \leq a\phi(\varepsilon) < \phi(\varepsilon)$ which is a contradiction.

This shows that $\{x_n\}$ is a Cauchy sequence, as X is complete, limit of $\{x_n\}$ exists. Then from (A3), we have

$$\phi(d(T_m y, x_{n(k)})) = \phi(d(T_m y, T_{n(k)} x_{n(k)-1}))$$

$$\leq a \max\{\phi(d(y, x_{n(k)-1})), \phi(d(y, y)), \phi(d(x_{n(k)}, x_{n(k)-1})),$$
$$\phi(d(y, x_{n(k)-1})), \phi(d(x_{n(k)-1}, y))\} + b(\phi(d(T_m y, y)) + \phi(d(x_{n(k)}, x_{n(k)-1}))).$$

Taking limit as $k \to \infty, \phi(d(T_m y, y)) \leq b\phi(d(T_m y, y))$.
Hence, $\phi(d(T_m y, y)) = 0$ i.e., $d(T_m y, y) = 0$.

So, y is a fixed point of T_m. Thus, y is a common fixed point for the sequence $\{T_n\}_{n=1}^{\infty}$.
Uniqueness of the fixed point follows from (A3)
The following example shows the applicability of Theorem 2.2 with $\phi(t) = t^2$.

Example 2.1

Let $X = [0, 0.1]$ with usual metric.
Define $T_n : X \to X$ by $T_n x = x^{2n}$ for $n = 1, 2, ...$
Define $\phi(t) = t^2 \quad , t \geq 0$ so that $\phi \in \Phi$.
Let $x, y \in X, \quad x \neq y$.
Then

$$\phi(d(T_n x, T_m y)) = (x^{2n} - y^{2m})^2 \leq (0.04)(x^n - x^m)^2 \leq (0.04)(x^{2n} + y^{2m}) \tag{18}$$

and

$$(0.04)(x^{2n} + y^{2m}) + 2(0.05)(x^{2n+1} + y^{2m+1}) + 2(0.03)(yx^{2n} + xy^{2m})$$
$$= [0.04 + (0.1)x]x^{2n} + [0.04 + (0.1)y]y^{2m} + 0.06xy(x^{2n-1} + y^{2m-1})$$
$$\leq (0.05)(x^{2n} + y^{2m}) + 0.03(x^{2n-1} + y^{2m-1})$$
$$\leq (0.01)(x-y)^2 + (0.05)(x^2+y^2) + (0.05)(x^{4n}+y^{4n}) + (0.03)[(x^2+y^2) + (x^{4n}+y^{4m})]$$

for all $m \geq 1$ and $n \geq 1$. As such we get

$$(0.04)(x^{2n} + y^{2m}) \leq (0.01)(x-y)^2 + (0.05)[(x - x^{2n})^2 + (y - y^{2m})^2]$$
$$+(0.03)[(x - y^{2m})^2 + (y - x^{2n})^2] \tag{19}$$

From (18) and (19), it follows that the inequality (A1) holds with $a = 0.01, b = 0.05$ and $c = 0.03$.

Condition (A2) holds trivially for any $0 \neq x_0 \in X$ and 0 is the unique common fixed point of $\{T_n\}_{n=1}^{\alpha}$.

References

[1] Jungck, G., Compatible mappings and common fixed points, Int. J. Math. & Math. Sci. 9(1986), 771-779.

[2] Khan, M. S., Swaleh, M., Sessa, S. , Fixed point theorem by altering distances between the points, Bull. Aust. Math. Soc. 30(1984), 1-9.

[3] Pant, R. P., Common fixed points of four mappings, Bull. Cal. Math. Soc. 90(1998), 281-286.

[4] - - - - - - - -, A common fixed point theorem under a new condition, Ind. J. Pure Appl. Math., 30(2) (1999), 147-152.

[5] Naidu, S. V. R., Fixed point theorem by altering distances, Adv. Math. Sci. Appl. 11(2001), 1-16.

[6] - - - - - - - -, Some fixed point theorems in metric spaces by altering distances, Czech. Math. Jour. 53(128) (2003), 205-212.

[7] Sastry, K. P. R. and Babu, G. V. R., Some fixed point theorems by altering distances between the points, Ind. Jour. Pure Appl. Math. 30(1999), 641-647.

[8] - - - - - - - -, A common fixed point theorem in complete metric spaces by altering distances, Proc.Nat.Acad.Sci.India , 71(A)-III,(2001), 237-242.