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On certain transformation formulae for terminating hypergeometric series

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Abstract: In this paper, making use of Bailey's Lemma and certain known summation formulae an attempt will be made to establish transformations involving terminating basic hypergeometric series. We shall deduce the transformations involving terminating and truncated series from our results

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1. Introduction, Notation and Definition

Throughout this paper we shall adopt the following notation and definition;
For any numbers a and q, real or complex and $|q| < 1$, let

$$[\alpha; q]_n \equiv [\alpha]_n = \begin{cases} (1-\alpha)(1-\alpha q)(1-\alpha q^2)...(1-\alpha q^{n-1}); & n > 0 \\ 1 \qquad ; & n = 0 \end{cases} \tag{1.1}$$

Accordingly, we have

$$[\alpha; q]_\infty = \prod_{n=0}^{\infty}(1-\alpha q^n)$$

Also,

$$[a_1, a_2, a_3, ..., a_r; q]_n \equiv [a_1; q]_n[a_2; q]_n[a_3; q]_n...[a_r; q]_n. \tag{1.2}$$

We define a basic hypergeometric series

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{array}\right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n q^{\lambda n(n+1)/2}}{[q, b_1, b_2, ..., b_s; q]_n} \quad (|z| < 1). \tag{1.3}$$

We also define a truncated series,

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{array}\right]_N = \sum_{n=0}^{N} \frac{[a_1, a_2, ..., a_r; q]_n z^n q^{\lambda n(n+1)/2}}{[q, b_1, b_2, ..., b_s; q]_n}, \tag{1.4}$$

Andrews [1] established the Bailey's Lemma in the following form,
If

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q; q]_{n-r}[aq; q]_{n+r}} \tag{1.4(a)}$$

then

$$\sum_{n=0}^{N} \frac{[\rho_1, \rho_2; q]_n (aq/\rho_1\rho_2)^n \alpha_n}{[aq/\rho_1, aq/\rho_2; q]_n [q; q]_{N-n}[aq; q]_{N+n}}$$

$$= \sum_{n=0}^{N} \frac{[\rho_1, \rho_2; q]_n [aq/\rho_1\rho_2; q]_{N-n} (aq/\rho_1\rho_2)^n \beta_n}{[q; q]_{N-n}[aq/\rho_1, aq/\rho_2; q]_N} \tag{1.4(b)}$$

Andrews [1; (2.3) and (2.4), p. 270]

We shall need the following known results in our analysis,

$${}_4\Phi_3\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, q^{-m}; q; -q^{-\frac{1}{2}+m} \\ \sqrt{a}, -\sqrt{a}, aq^{1+m} \end{array}\right]$$

$$= \frac{[aq, -q^{-\frac{1}{2}}; q]_m}{2[\sqrt{aq}; q]_m[-q\sqrt{a}; q]_{m-1}} + \frac{[aq, -q^{-\frac{1}{2}}; q]_m}{2[-\sqrt{aq}; q]_m[q\sqrt{a}; q]_{m-1}} \tag{1.5}$$

Verma and Jain [2; (4.1), p. 76]

$${}_3\Phi_2\left[\begin{array}{l} a, q\sqrt{a}, q^{-m}; q; -q^m \\ \sqrt{a}, aq^{m+1} \end{array}\right]$$

$$= \frac{(1+\sqrt{a})[aq, -1; q]_m}{2[aq; q^2]_m} + \frac{(1-\sqrt{a})[aq, -1; q]_m}{2[-\sqrt{a}, -q\sqrt{a}; q]_m} \tag{1.6}$$

Verma and Jain [2; (4.2), p. 76]

$$
{}_2\Phi_1\left[\begin{matrix} a, q^{-m}; q; -q^{\frac{1}{2}+m} \\ aq^{m+1} \end{matrix}\right]
= \frac{(1+\sqrt{a})[aq, -\sqrt{q}; q]_m}{2[-\sqrt{aq}, q\sqrt{a}; q]_m} + \frac{(1-\sqrt{a})[aq, -\sqrt{q}; q]_m}{2[\sqrt{aq}, -q\sqrt{a}; q]_m} \tag{1.7}
$$

Verma and Jain [2; (4.3), p. 76]

$$
{}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-m}; q; -q^{\frac{1}{2}+m} \\ \sqrt{a}, -\sqrt{a}, aq^{m+1} \end{matrix}\right]
= \frac{1}{2\sqrt{a}}\left\{\frac{[aq, -q^{-\frac{1}{2}}; q]_m}{[\sqrt{aq}; q]_m[-q\sqrt{a}; q]_{m-1}} - \frac{[aq, -q^{-\frac{1}{2}}; q]_m}{[\sqrt{-aq}; q]_m[q\sqrt{a}; q]_{m-1}}\right\} \tag{1.8}
$$

Verma and Jain [2; (4.5), p. 77]

$$
{}_3\Phi_2\left[\begin{matrix} a, q\sqrt{a}, q^{-m}; q; -q^{m+1} \\ \sqrt{a}, aq^{m+1} \end{matrix}\right]
= \frac{1+\sqrt{a}}{2\sqrt{a}}\frac{[aq, -1; q]_m}{[aq; q^2]_m} - \frac{1-\sqrt{a}}{2\sqrt{a}}\frac{[aq, -1; q]_m}{[\sqrt{a}, -q\sqrt{a}; q]_m} \tag{1.9}
$$

Verma and Jain [2; (4.6), p. 77]

$$
{}_2\Phi_1\left[\begin{matrix} a, q^{-m}; q; -q^{\frac{3}{2}+m} \\ aq^{m+1} \end{matrix}\right]
= \frac{1+\sqrt{a}}{2\sqrt{a}}\frac{[aq, -\sqrt{q}; q]_m}{[\sqrt{aq}, q\sqrt{a}; q]_m} - \frac{1-\sqrt{a}}{2\sqrt{a}}\frac{[aq, -\sqrt{q}; q]_m}{[\sqrt{aq}, -q\sqrt{a}; q]_m} \tag{1.10}
$$

Verma and Jain [2; (4.7), p. 77]

2. Transformations involving terminating series

In this section we shall establish our main transformations. We shall also deduce transformations involving truncated series from our results.

If we set

$$
\alpha_r = \frac{q^{r(r-1)/2}[a, q\sqrt{a}, -q\sqrt{a}; q]_r z^r}{[q, \sqrt{a}, -\sqrt{a}; q]_r}
$$

in (1.4(a)), we get

$$
\beta_n = \frac{1}{[q, aq; q]_n}\, {}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; q; -zq^n \\ \sqrt{a}, -\sqrt{a}, aq^{n+1} \end{matrix}\right] \tag{2.1}
$$

Now putting $z = q^{-1/2}$ in (2.1) and using (1.5), we get

$$\beta_n = \frac{1}{[q, aq; q]_n} \left\{ \frac{(1+\sqrt{a})[aq, -q^{-\frac{1}{2}}; q]_n}{2[\sqrt{aq}, -\sqrt{a}; q]_n} + \frac{(1-\sqrt{a})[aq, -q^{-\frac{1}{2}}; q]_n}{2[-\sqrt{aq}, -\sqrt{a}; q]_n} \right\} \tag{2.2}$$

Substituting the above values of α_n and β_n in (1.4(b)), we get after some simplifications

$$\frac{[aq/\rho_1, aq/\rho_2; q]_N}{[aq, aq/\rho_1\rho_2; q]_N} \, {}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \rho_1, \rho_2, q^{-N}; q; -aq^{N+\frac{1}{2}}/\rho_1\rho_2 \\ \sqrt{a}, -\sqrt{a}, aq/\rho_1, aq/\rho_2, aq^{N+1} \end{matrix} \right]$$

$$= \frac{1+\sqrt{a}}{2} \, {}_4\Phi_3 \left[\begin{matrix} \rho_1, \rho_2, -q^{-\frac{1}{2}}, q^{-N}; q; q \\ \sqrt{aq}, -\sqrt{a}, \rho_1\rho_2 q^{-N}/a \end{matrix} \right]$$

$$+ \frac{1-\sqrt{a}}{2} \, {}_4\Phi_3 \left[\begin{matrix} \rho_1, \rho_2, -q^{-\frac{1}{2}}, q^{-N}; q; q \\ -\sqrt{aq}, \sqrt{a}, \rho_1\rho_2 q^{-N}/a \end{matrix} \right] \tag{2.3}$$

Next, taking

$$\alpha_r = \frac{q^{r(r-1)/2}[a, q\sqrt{a}; q]_r z^r}{[q, \sqrt{a}; q]_r}$$

in (1.4(a)), we get

$$\beta_n = \frac{1}{[q, aq; q]_n} \, {}_3\Phi_2 \left[\begin{matrix} a, q\sqrt{a}, q^{-n}; q; -zq^n \\ \sqrt{a}, aq^{n+1} \end{matrix} \right] \tag{2.4}$$

Now putting $z = 1$ in (2.4) and using (1.6) to sum the ${}_3\Phi_2$, we get

$$\beta_n = \frac{1}{[q; q]_n} \left\{ \frac{(1+\sqrt{a})[-1; q]_n}{2[aq; q^2]_n} + \frac{(1-\sqrt{a})[-1; q]_n}{[\sqrt{a}, -q\sqrt{a}; q]_n} \right\}$$

Now, with above values of α_n and β_n in (1.4(b)), leads to

$$\frac{[aq/\rho_1, aq/\rho_2; q]_N}{[aq, aq/\rho_1\rho_2; q]_N} \, {}_5\Phi_4 \left[\begin{matrix} a, q\sqrt{a}, \rho_1, \rho_2, q^{-N}; q; -aq^{N+1}/\rho_1\rho_2 \\ \sqrt{a}, aq/\rho_1, aq/\rho_2, aq^{N+1} \end{matrix} \right]$$

$$= \frac{1+\sqrt{a}}{2} \, {}_4\Phi_3 \left[\begin{matrix} \rho_1, \rho_2, -1, q^{-N}; q; q \\ \sqrt{aq}, -\sqrt{aq}, \rho_1\rho_2 q^{-N}/a \end{matrix} \right]$$

$$+ \frac{1-\sqrt{a}}{2} \, {}_4\Phi_3 \left[\begin{matrix} \rho_1, \rho_2, -1, q^{-N}; q; q \\ \sqrt{a}, -q\sqrt{a}, \rho_1\rho_2 q^{-N}/a \end{matrix} \right] \tag{2.5}$$

Further putting

$$\alpha_r = \frac{q^{r(r-1)/2}[a;q]_r z^r}{[q;q]_r}$$

in (1.4(a)), we get

$$\beta_n = \frac{1}{[q,aq;q]_n}\ {}_2\Phi_1\left[\begin{matrix} a, q^{-n}; q; -zq^n \\ aq^{n+1} \end{matrix}\right] \tag{2.6}$$

Now, setting $z = q^{1/2}$ in (2.6) and summing the ${}_2\Phi_1$ on the right with the help of (1.7), we get,

$$\beta_n = \frac{1}{[q;q]_n}\left\{\frac{(1+\sqrt{a})[-\sqrt{q};q]_n}{2[-\sqrt{aq},q\sqrt{a};q]_n} + \frac{(1-\sqrt{a})[-\sqrt{q};q]_n}{2[\sqrt{aq},-q\sqrt{a};q]_n}\right\} \tag{2.7}$$

Substituting the above values of α_n and β_n in (1.4(b)), we get after some simplification

$$\begin{aligned}
&\frac{[aq/\rho_1, aq/\rho_2;q]_N}{[aq, aq/\rho_1\rho_2;q]_N}\ {}_4\Phi_3\left[\begin{matrix} a, \rho_1, \rho_2, q^{-N}; q; -aq^{\frac{3}{2}+N}/\rho_1\rho_2 \\ aq/\rho_1, aq/\rho_2, aq^{N+1} \end{matrix}\right] \\
&= \frac{1+\sqrt{a}}{2}\ {}_4\Phi_3\left[\begin{matrix} \rho_1, \rho_2, -\sqrt{q}, q^{-N}; q; q \\ -\sqrt{aq}, -q\sqrt{a}, \rho_1\rho_2 q^{-N}/a \end{matrix}\right] \\
&+ \frac{1-\sqrt{a}}{2}\ {}_4\Phi_3\left[\begin{matrix} \rho_1, \rho_2, -\sqrt{q}, q^{-N}; q; q \\ \sqrt{aq}, -q\sqrt{a}, \rho_1\rho_2 q^{-N}/a \end{matrix}\right]
\end{aligned} \tag{2.8}$$

Next, taking

$$\alpha_r = \frac{q^{r(r-1)/2}[a, q\sqrt{a}, -q\sqrt{a};q]_r z^r}{[q,\sqrt{a},-\sqrt{a};q]_r}$$

in (1.4(a)) and using (1.8) with $z = \sqrt{q}$, we get after some simplification

$$\beta_n = \frac{1}{[q;q]_n}\left\{\frac{(1+\sqrt{a})[-q^{-1/2};q]_n}{2\sqrt{a}[\sqrt{aq},-\sqrt{a};q]_n} - \frac{(1-\sqrt{a})[-q^{-1/2};q]_n}{2\sqrt{a}[-\sqrt{aq},\sqrt{a};q]_n}\right\} \tag{2.9}$$

Now, with the above values of α_n and β_n in (1.4(b)), we get

$$\begin{aligned}
&\frac{[aq/\rho_1, aq/\rho_2;q]_N}{[aq, aq/\rho_1\rho_2;q]_N}\ {}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \rho_1, \rho_2, q^{-N}; q; -aq^{\frac{3}{2}+N}/\rho_1\rho_2 \\ \sqrt{a}, -\sqrt{a}, aq/\rho_1, aq/\rho_2, aq^{N+1} \end{matrix}\right] \\
&= \frac{(1+\sqrt{a})}{2\sqrt{a}}\ {}_4\Phi_3\left[\begin{matrix} \rho_1, \rho_2, -q^{-1/2}, q^{-N}; q; q \\ \sqrt{aq}, -\sqrt{a}, \rho_1\rho_2 q^{-N}/a \end{matrix}\right]
\end{aligned}$$

$$-\frac{(1-\sqrt{a})}{2\sqrt{a}}\ {}_4\Phi_3\left[\begin{array}{l}\rho_1,\rho_2,-q^{-1/2},q^{-N};q;q\\ -\sqrt{aq},\sqrt{a},\rho_1\rho_2 q^{-N}/a\end{array}\right] \tag{2.10}$$

Again with

$$\alpha_r=\frac{q^{r(r-1)/2}[a,q\sqrt{a};q]_r z^r}{[q,\sqrt{a};q]_r}$$

in (1.4(a)), we get

$$\beta_n=\frac{1}{[q,aq;q]_n}\ {}_3\Phi_2\left[\begin{array}{l}a,q\sqrt{a},q^{-n};q;-zq^n\\ \sqrt{a},aq^{n+1}\end{array}\right] \tag{2.11}$$

Now, taking $z=q$ in (2.11) and summing the ${}_3\Phi_2$ with the help of (1.9) and substituting the resulting value of β_n and the above value of α_n, we get

$$\frac{[aq/\rho_1,aq/\rho_2;q]_N}{[aq,aq/\rho_1\rho_2;q]_N}\ {}_5\Phi_4\left[\begin{array}{l}a,q\sqrt{a},\rho_1,\rho_2,q^{-N};q;-aq^{2+N}/\rho_1\rho_2\\ \sqrt{a},aq/\rho_1,aq/\rho_2,aq^{N+1}\end{array}\right]$$

$$=\frac{(1+\sqrt{a})}{2\sqrt{a}}\ {}_4\Phi_3\left[\begin{array}{l}\rho_1,\rho_2,-1,q^{-N};q;q\\ \sqrt{aq},-\sqrt{aq},\rho_1\rho_2 q^{-N}/a\end{array}\right]$$

$$-\frac{(1-\sqrt{a})}{2\sqrt{a}}\ {}_4\Phi_3\left[\begin{array}{l}\rho_1,\rho_2,-1,q^{-N};q;q\\ \sqrt{a},-q\sqrt{a},\rho_1\rho_2 q^{-N}/a\end{array}\right] \tag{2.12}$$

Further, if we set

$$\alpha_r=\frac{q^{r(r-1)/2}[a;q]_r z^r}{[q;q]_r}$$

in (1.4(a)), we get

$$\beta_n=\frac{1}{[q,aq;q]_n}\ {}_2\Phi_1\left[\begin{array}{l}a,q^{-n};q;-zq^n\\ aq^{n+1}\end{array}\right] \tag{2.13}$$

Putting $z=q^{3/2}$ in (2.13) and using (1.10), to sum the ${}_2\Phi_1$, we get

$$\beta_n=\frac{1}{[q,aq;q]_n}\left\{\frac{(1+\sqrt{a})[aq,-\sqrt{q};q]_n}{2\sqrt{a}[-\sqrt{aq},q\sqrt{a};q]_n}-\frac{(1-\sqrt{a})[aq,-\sqrt{q};q]_n}{2\sqrt{a}[\sqrt{aq},-q\sqrt{a};q]_n}\right\}$$

Substituting the above values of α_n and β_n, we get

$$\frac{[aq/\rho_1,aq/\rho_2;q]_N}{[aq,aq/\rho_1\rho_2;q]_N}\ {}_4\Phi_3\left[\begin{array}{l}a,\rho_1,\rho_2,q^{-N};q;-aq^{\frac{5}{2}+N}/\rho_1\rho_2\\ aq/\rho_1,aq/\rho_2,aq^{N+1}\end{array}\right]$$

$$= \frac{(1+\sqrt{a})}{2\sqrt{a}}\ {}_4\Phi_3\left[\begin{matrix} \rho_1, \rho_2, -\sqrt{q}, q^{-N}; q; q \\ -\sqrt{aq}, q\sqrt{a}, \rho_1\rho_2 q^{-N}/a \end{matrix}\right]$$

$$-\frac{(1-\sqrt{a})}{2\sqrt{a}}\ {}_4\Phi_3\left[\begin{matrix} \rho_1, \rho_2, -\sqrt{q}, q^{-N}; q; q \\ \sqrt{aq}, -q\sqrt{a}, \rho_1\rho_2 q^{-N}/a \end{matrix}\right] \tag{2.14}$$

If we replace a by a/q in (1.4(a)) and (1.4(b)), we get the following form of Bailey's Lemma;
If for $n \geq 0$,

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q;q]_{n-r}[a;q]_{n+r}} \tag{2.15}$$

then,

$$\sum_{n=0}^{N} \frac{[\rho_1, \rho_2; q]_n (a/\rho_1\rho_2)^n \alpha_n}{[a/\rho_1, a/\rho_2; q]_n [q;q]_{N-n}[q;q]_{N+n}}$$

$$= \sum_{n=0}^{N} \frac{[\rho_1, \rho_2; q]_n [a/\rho_1\rho_2; q]_{N-n} (a/\rho_1\rho_2)^n \beta_n}{[q;q]_{N-n}[a/\rho_1, a/\rho_2; q]_N} \tag{2.16}$$

We shall use the above two results (2.15) and (2.16) to establish certain transformations involving terminating series.
If we set

$$\alpha_r = \frac{q^{r(r-1)/2}[a, q\sqrt{a}, -q\sqrt{a}; q]_r z^r}{[q, \sqrt{a}, -\sqrt{a}; q]_r}$$

in (2.15), we get

$$\beta_n = \frac{1}{[q, a; q]_n}\ {}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; q; -zq^n \\ \sqrt{a}, -\sqrt{a}, aq^n \end{matrix}\right] \tag{2.17}$$

Now, taking $z = q^{-1/2}$ in the above and summing the ${}_4\Phi_3$ with the help of the following known summation due to Verma and Jain [2; (4.15), p.80],

$${}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-m}; q; -q^{\frac{1}{2}+m} \\ \sqrt{a}, -\sqrt{a}, aq^m \end{matrix}\right]$$

$$= \frac{[a, q\sqrt{a}, -q^{-1/2}; q]_m}{2[\sqrt{a}, \sqrt{aq}, -\sqrt{a}; q]_m} + \frac{[a, -q\sqrt{a}, -q^{-1/2}; q]_m}{2[-\sqrt{a}, -\sqrt{aq}, \sqrt{a}; q]_m}$$

we get

$$\beta_n = \frac{1}{2[q;q]_n}\left\{\frac{[q\sqrt{a}, -q^{-1/2}; q]_n}{[\sqrt{a}, \sqrt{aq}, -\sqrt{a}; q]_n} + \frac{[-q\sqrt{a}, -q^{-1/2}; q]_n}{[-\sqrt{a}, -\sqrt{aq}, \sqrt{a}; q]_n}\right\}$$

Substituting the above values of α_n and β_n in (2.16), we get

$$\frac{2[a/\rho_1, a/\rho_2; q]_N}{[a, a/\rho_1\rho_2; q]_N} {}_6\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, \rho_1, \rho_2, q^{-N}; q; -aq^{N-\frac{1}{2}}/\rho_1\rho_2 \\ a/\rho_1, a/\rho_2, \sqrt{a}, -\sqrt{a}, aq^N \end{array}\right]$$

$$= {}_5\Phi_4\left[\begin{array}{l} q\sqrt{a}, -q^{-1/2}, \rho_1, \rho_2, q^{-N}; q; q \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, \rho_1\rho_2 q^{1-N}/a \end{array}\right]$$

$$+ {}_5\Phi_4\left[\begin{array}{l} -q\sqrt{a}, -q^{-1/2}, \rho_1, \rho_2, q^{-N}; q; q \\ \sqrt{a}, -\sqrt{a}, -\sqrt{aq}, \rho_1\rho_2 q^{1-N}/a \end{array}\right] \tag{2.18}$$

Next, if we put

$$\alpha_r = \frac{q^{r(r-1)/2}[a, q\sqrt{a}; q]_r z^r}{[q, \sqrt{a}; q]_r}$$

in (2.15), we get

$$\beta_n = \frac{1}{[q, a; q]_n} {}_3\Phi_2\left[\begin{array}{l} a, q\sqrt{a}, q^{-n}; q; -zq^n \\ \sqrt{a}, aq^n \end{array}\right]$$

Putting $z = 1$ in the above and summing the ${}_3\Phi_2$ with the help of the following result due to Verma and Jain [2; (4.16), p.80],

$${}_3\Phi_2\left[\begin{array}{l} a, q\sqrt{a}, q^{-m}; q; -q^m \\ \sqrt{a}, aq^m \end{array}\right]$$

$$= \frac{[a, -1, q\sqrt{a}; q]_m}{2[\sqrt{a}, q\sqrt{aq}, -\sqrt{aq}; q]_m} + \frac{[a, -1; q]_m}{2[\sqrt{a}, -\sqrt{a}; q]_m}$$

we get

$$\beta_n = \frac{1}{2[q; q]_n}\left\{\frac{[q\sqrt{a}, -1; q]_n}{[\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q]_n} + \frac{[-1; q]_n}{[\sqrt{a}, -\sqrt{a}; q]_n}\right\}$$

Now, substituting the above values of α_n and β_n in (2.16), we get

$$\frac{2[a/\rho_1, a/\rho_2; q]_N}{[a, a/\rho_1\rho_2; q]_N} {}_5\Phi_4\left[\begin{array}{l} a, q\sqrt{a}, \rho_1, \rho_2, q^{-N}; q; -aq^N/\rho_1\rho_2 \\ a/\rho_1, a/\rho_2, \sqrt{a}, aq^N \end{array}\right]$$

$$= {}_5\Phi_4\left[\begin{array}{l} q\sqrt{a}, \rho_1, \rho_2, -1, q^{-N}; q; q \\ \sqrt{a}, \sqrt{aq}, -\sqrt{aq}, \rho_1\rho_2 q^{1-N}/a \end{array}\right]$$

$$+ {}_4\Phi_3\left[\begin{array}{l} \rho_1, \rho_2, -1, q^{-N}; q; q \\ \sqrt{a}, -\sqrt{a}, \rho_1\rho_2 q^{1-N}/a \end{array}\right] \tag{2.19}$$

Further, setting

$$\alpha_r = \frac{q^{r^2/2}[a;q]_r z^r}{[q;q]_r}$$

in (2.15) and making use of yet another known result due to Verma and Jain [2; (4.17), p. 80], namely

$${}_2\Phi_1\left[\begin{array}{l} a, q^{-m}; q; -q^{m+\frac{1}{2}} \\ aq^m \end{array}\right] = \frac{[a, -\sqrt{q}; q]_m}{2[-\sqrt{aq}, \sqrt{a}; q]_m} + \frac{[a, -\sqrt{q}; q]_m}{2[\sqrt{aq}, -\sqrt{a}; q]_m}$$

we get

$$\beta_n = \frac{1}{2[q;q]_n}\left\{\frac{[\sqrt{q};q]_n}{[\sqrt{a}, -\sqrt{aq}; q]_n} + \frac{[-\sqrt{q};q]_n}{[-\sqrt{a}, \sqrt{aq}; q]_n}\right\}$$

Substituting the above values of α_n and β_n in (2.16), we get

$$\frac{2[a/\rho_1, a/\rho_2; q]_N}{[a, a/\rho_1\rho_2; q]_N}\ {}_4\Phi_3\left[\begin{array}{l} a, \rho_1, \rho_2, q^{-N}; q; -aq^{N+\frac{1}{2}}/\rho_1\rho_2 \\ a/\rho_1, a/\rho_2, aq^N \end{array}\right]$$

$$= {}_4\Phi_3\left[\begin{array}{l} \rho_1, \rho_2, -\sqrt{q}, q^{-N}; q; q \\ \sqrt{a}, -\sqrt{aq}, \rho_1\rho_2 q^{1-N}/a \end{array}\right]$$

$$+ {}_4\Phi_3\left[\begin{array}{l} \rho_1, \rho_2, -\sqrt{q}, q^{-N}; q; q \\ -\sqrt{a}, \sqrt{aq}, \rho_1\rho_2 q^{1-N}/a \end{array}\right] \tag{2.20}$$

Now, if we replace a by aq in (1.4(a)) and (1.4(b)), we get
If, for $n \geq 0$,

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{[q;q]_{n-r}[aq^2;q]_{n+r}} \tag{2.21}$$

then

$$\sum_{n=0}^{N} \frac{[\rho_1, \rho_2; q]_n (aq^2/\rho_1\rho_2)^n \alpha_n}{[aq^2/\rho_1, aq^2/\rho_2; q]_n [q;q]_{N-n}[aq^2;q]_{N+n}}$$

$$= \sum_{n=0}^{N} \frac{[\rho_1, \rho_2; q]_n [aq^2/\rho_1\rho_2; q]_{N-n} (aq^2/\rho_1\rho_2)^n \beta_n}{[q;q]_{N-n}[aq^2/\rho_1, aq^2/\rho_2; q]_N} \tag{2.22}$$

Now, if we put

$$\alpha_r = \frac{q^{r(r+1)/2}[a, q\sqrt{a}; q]_r}{[q, \sqrt{a}; q]_r}$$

and make use of the following known summation due to Verma and Jain [2; (4.8), p.79]

$$ {}_3\Phi_2\left[\begin{array}{l} a, q\sqrt{a}, q^{-m}; q; -q^{m+1} \\ \sqrt{a}, aq^{m+2} \end{array}\right] $$

$$ = \frac{[a;q]_{m+2}[q\sqrt{a}, -1;q]_{m+1}[q;q]_m}{2\sqrt{a}[q, -\sqrt{aq}, \sqrt{a}, \sqrt{aq};q]_{m+1}} - \frac{[a;q]_{m+2}[-1;q]_{m+1}[q;q]_m}{2\sqrt{a}[\sqrt{a}, -\sqrt{a}, q;q]_{m+1}} $$

to calcultae the value of β_n and the substitute this value of β_n and the above value of α_n in (2.22), we get

$$ \frac{[aq^2/\rho_1, aq^2/\rho_2;q]_N}{[aq^2, aq^2/\rho_1\rho_2;q]_N} {}_5\Phi_4\left[\begin{array}{l} a, q\sqrt{a}, \rho_1, \rho_2, q^{-N}; q; -aq^{3+N}/\rho_1\rho_2 \\ \sqrt{a}, aq^2/\rho_1, aq^2/\rho_2, aq^{N+2} \end{array}\right] $$

$$ = \frac{(1-q\sqrt{a})(1+\sqrt{a})}{2\sqrt{a}(1-q)} {}_6\Phi_5\left[\begin{array}{l} \rho_1, \rho_2, q, -q, q^2\sqrt{a}, q^{-N}; q; q \\ \rho_1\rho_2 q^{-N-1}/a, q^2, q\sqrt{a}, q\sqrt{aq}, -q\sqrt{aq} \end{array}\right] $$

$$ -\frac{(1-aq)}{2\sqrt{a}(1-q)} {}_5\Phi_4\left[\begin{array}{l} \rho_1, \rho_2, q, -q, q^{-N}; q; q \\ \rho_1\rho_2 q^{-N-1}/a, q^2, -q\sqrt{a}, q\sqrt{a} \end{array}\right] \tag{2.23} $$

Further, if we put

$$ \alpha_r = \frac{q^{r^2/2}[a, q\sqrt{a}, -\sqrt{a};q]_r}{[q, \sqrt{a}, -\sqrt{a};q]_r} $$

in (2.21) and use the following known summation due to Verma and Jain [2; (4.9), p. 78]

$$ {}_4\Phi_3\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, q^{-m}; q; -q^{m+\frac{1}{2}} \\ \sqrt{a}, -\sqrt{a}, aq^{2+m} \end{array}\right] $$

$$ = \frac{[a;q]_{m+2}[q\sqrt{a}, -q^{-1/2};q]_{m+1}[q;q]_m}{2\sqrt{a}[q, \sqrt{a}, -\sqrt{a}, \sqrt{aq};q]_{m+1}} $$

$$ -\frac{[a;q]_{m+2}[-q\sqrt{a}, -q^{-1/2};q]_{m+1}[q;q]_m}{2\sqrt{a}[q, \sqrt{a}, -\sqrt{a}, -\sqrt{aq};q]_{m+1}} \tag{2.24} $$

for the evaluation of β_n, we get after proper substitution for α_n and β_n in (2.22)

$$ \frac{[aq^2/\rho_1, aq^2/\rho_2;q]_N}{[aq^2, aq^2/\rho_1\rho_2;q]_N} {}_6\Phi_5\left[\begin{array}{l} \rho_1, \rho_2, a, q\sqrt{a}, -q\sqrt{a}, q^{-N}; q; -aq^{\frac{5}{2}+N}/\rho_1\rho_2 \\ aq^2/\rho_1, aq^2/\rho_2, \sqrt{a}, -\sqrt{a}, aq^{2+N} \end{array}\right] $$

$$ = \frac{(1+\sqrt{aq})(1+q\sqrt{a})}{2\sqrt{aq}(1-\sqrt{q})} {}_6\Phi_5\left[\begin{array}{l} \rho_1, \rho_2, -q^2\sqrt{a}, -q^{-1/2}, q, q^{-N}; q; q \\ \rho_1\rho_2 q^{-N-1}/a, q^2, q\sqrt{a}, -q\sqrt{a}, q\sqrt{aq} \end{array}\right] $$

$$-\frac{(1-\sqrt{aq})(1+q\sqrt{a})}{2\sqrt{aq}(1-\sqrt{q})}\ {}_6\Phi_5\left[\begin{matrix}\rho_1,\rho_2,-q^2\sqrt{a},-q^{-1/2},q,q^{-N};q;q\\ \rho_1\rho_2q^{-N-1}/a,q^2,q\sqrt{a},-q\sqrt{a},-q\sqrt{aq}\end{matrix}\right] \tag{2.25}$$

Finally, if we set

$$\alpha_r=\frac{q^{r(r+2)/2}[a;q]_r}{[q;q]_r}$$

in (2.21) and use the following known summation due to Verma and Jain [2; (4.11), p. 79], namely

$${}_2\Phi_1\left[\begin{matrix}a,q^{-m};q;-q^{m+\frac{3}{2}}\\ aq^{2+m}\end{matrix}\right]$$

$$=\frac{[a;q]_{m+2}[-\sqrt{q};q]_{m+1}[q;q]_m}{2\sqrt{a}[q,\sqrt{a},-\sqrt{aq};q]_{m+1}}-\frac{[a;q]_{m+2}[-\sqrt{q};q]_{m+1}[q;q]_m}{2\sqrt{a}[q,-\sqrt{a},\sqrt{aq};q]_{m+1}}$$

to evaluate β_n, we get

$$\beta_n=\frac{1}{[q;q]_n}\left\{\frac{(1-a)(1-aq)(1+\sqrt{q})[q,-q^{3/2};q]_n}{2\sqrt{a}(1-\sqrt{a})(1-q)(1+\sqrt{aq})[q\sqrt{a},q^2,-q\sqrt{aq};q]_n}\right.$$

$$\left.-\frac{(1-a)(1-aq)(1+\sqrt{q})[q,-q^{3/2};q]_n}{2\sqrt{a}(1-\sqrt{aq})(1-q)(1+\sqrt{a})[-q\sqrt{a},q^2,q\sqrt{aq};q]_n}\right\}$$

Substituting the above values of α_n and β_n in (2.22), we get

$$\frac{[aq^2/\rho_1,aq^2/\rho_2;q]_N}{[aq^2,aq^2/\rho_1\rho_2;q]_N}\ {}_4\Phi_3\left[\begin{matrix}\rho_1,\rho_2,a,q^{-N};q;-aq^{\frac{7}{2}+N}/\rho_1\rho_2\\ aq^2/\rho_1,aq^2/\rho_2,aq^{2+N}\end{matrix}\right]$$

$$=\frac{(1-a)(1-aq)(1+\sqrt{q})}{2\sqrt{a}(1-q)(1-\sqrt{aq})(1-\sqrt{a})}\ {}_5\Phi_4\left[\begin{matrix}\rho_1,\rho_2,q^{-N},q,-q^{3/2};q;q\\ \rho_1\rho_2q^{-N-1}/a,q^2,q\sqrt{a},-q\sqrt{aq}\end{matrix}\right]$$

$$-\frac{(1-a)(1-aq)(1+\sqrt{q})}{2\sqrt{a}(1-q)(1-\sqrt{aq})(1+\sqrt{a})}\ {}_5\Phi_4\left[\begin{matrix}\rho_1,\rho_2,q^{-N},q,-q^{3/2};q;q\\ \rho_1\rho_2q^{-N-1}/a,q^2,-q\sqrt{a},q\sqrt{aq}\end{matrix}\right] \tag{2.26}$$

The transformations investigated here involve terminating series on both sides. From the above results we can deduce the transformations in which both the series on the right are truncated. We sight the following result.
If we let $a\to\rho_1\rho_2$ in (2.3), we get

$$\frac{[q\rho_1,q\rho_2;q]_N}{[q\rho_1\rho_2,q;q]_N}\ {}_6\Phi_5\left[\begin{matrix}\rho_1\rho_2,q\sqrt{\rho_1\rho_2},-q\sqrt{\rho_1\rho_2},\rho_1,\rho_2,q^{-N};q;-q^{N+\frac{1}{2}}\\ \sqrt{\rho_1\rho_2},-\sqrt{\rho_1\rho_2},q\rho_1,q\rho_2,\rho_1\rho_2q^{N+1}\end{matrix}\right]$$

$$= \frac{1+\sqrt{\rho_1\rho_2}}{2}\ {}_3\Phi_2\left[\begin{array}{c}\rho_1, \rho_2, -q^{-1/2}; q; q\\ \sqrt{\rho_1\rho_2 q}, -\sqrt{\rho_1\rho_2}\end{array}\right]_N$$

$$+\frac{1-\sqrt{\rho_1\rho_2}}{2}\ {}_3\Phi_2\left[\begin{array}{c}\rho_1, \rho_2, -q^{-1/2}; q; q\\ -\sqrt{\rho_1\rho_2 q}, \sqrt{\rho_1\rho_2}\end{array}\right]_N \quad (2.27)$$

References

[1] Andrews, G.E., Multiple series Rogers-Ramanujan type identities, Pacific J. Math, Vol. 114, No. 2 (1984), 267-283.

[2] Verma and Jain, V.K., Certain summation formulae for q-series, Jour. Indian Math. Soc. 47 (1983), p. 71-85.