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On Dual WP Bailey Pairs and its Applications

Bindu Prakash Mishra
Department of Mathematics,
M.D. College, Parel, Mumbai-12 India
E-mail:- bindu1962@gmail.com

Abstract: In this paper, we have established certain transformation formulae for q-series by making use of dual WP-Bailey pairs.
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1. Introduction, Notations and Definitions

For $q, \lambda, \mu \in C \ \ (|q| < 1)$, the basic (or q-) shifted factorial $(\lambda; q)_\mu$ is defined by

$$(\lambda; q)_\mu = \prod_{i=0}^{\infty} \frac{(1-\lambda q^i)}{(1-\lambda q^{\mu+i})}, \tag{1.1}$$

so that

$$(\lambda; q)_n = \begin{cases} 1, & (n = 0) \\ (1-\lambda)(1-\lambda q)...(1-\lambda q^{n-1}), & (n \in N) \end{cases} \tag{1.2}$$

and

$$(\lambda; q)_\infty = \prod_{i=0}^{\infty} (1-\lambda q^i). \tag{1.3}$$

For convenience, we write

$$(a_1, a_2, ..., a_r; q)_n = (a_1; q)_n (a_2; q)_n ... (a_r; q)_n \tag{1.4}$$

and

$$(a_1, a_2, ..., a_r; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty ... (a_r; q)_\infty \tag{1.5}$$

The basic (or q-) hypergeometric function ${}_r\Phi_s$ is defined by,

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix} \right] = \sum_{n=0}^{\infty} (-1)^{n(1+s-r)} q^{(1+s-r)n(n-1)/2}$$

$$\times \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n}, \tag{1.6}$$

provided that the series in (1.6) is convergent.

In the study of Rogers-Ramanujan type identities Bailey [1] was led to the following simple, but remarkable observation.
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.7}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \tag{1.8}$$

Then, under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.9}$$

where u_n, v_n, α_n and δ_n are arbitrarily choosen sequences of n alone. In an application Bailey choose $u_r = \frac{1}{(q;q)_r}$ and $v_r = \frac{1}{(aq;q)_r}$ and derived the following result.
If

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \tag{1.10}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{(q;q)_r(aq;q)_{r+2n}} \tag{1.11}$$

Then, under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.12}$$

where α_n and δ_n are arbitrarily choosen sequences of n alone.

A pair of sequences $\langle \alpha_n, \beta_n \rangle$ that satisfy (1.10) are called Bailey pair with respect to the parameter a. Also, a pair of sequences $\langle \gamma_n, \delta_n \rangle$ satisfying (1.11) are called conjugate Bailey pair with respect to the parameter a.

Andrews work on the WP Bailey lemma is stated as,
If we choose $u_r = \frac{(k/a;q)_r}{(q;q)_r}$ and $v_r = \frac{(k;q)_r}{(aq;q)_r}$ in (1.7), we find the following result;

If

$$\beta_n(a,k,q) = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}} \alpha_r(a,k,q) \tag{1.13}$$

and

$$\gamma_n(a,k,q) = \sum_{r=0}^{\infty} \frac{(k/a;q)_{r}(k;q)_{r+2n}}{(q;q)_{r}(aq;q)_{r+2n}} \delta_{r+n}(a,k,q) \tag{1.14}$$

Then, under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n(a,k,q)\gamma_n(a,k,q) = \sum_{n=0}^{\infty} \beta_n(a,k,q)\delta_n(a,k,q), \tag{1.15}$$

where $\alpha_n(a,k,q)$ and $\delta_n(a,k,q)$ are arbitrarily choosen sequences.

A WP Bailey pair is a pair of sequences $\langle \alpha_n(a,k,q), \beta_n(a,k,q) \rangle$ satisfying (1.13) where as a conjugate WP Bailey pair is a pair of sequences $\langle \gamma_n(a,k,q), \delta_n(a,k,q) \rangle$ satisfying (1.14).

Dual of WP Bailey pair $\langle \alpha_n(a,k,q), \beta_n(a,k,q) \rangle$ is given by $\langle \alpha'_n(a,k,q), \beta'_n(a,k,q) \rangle$ where

$$\alpha'_n(a,k,q) = \alpha_n\left(\frac{1}{a}, \frac{1}{k}, \frac{1}{q}\right) \tag{1.16}$$

and

$$\beta'_n(a,k,q) = \left(\frac{k}{aq}\right)^{2n} \beta_n\left(\frac{1}{a}, \frac{1}{k}, \frac{1}{q}\right). \tag{1.17}$$

[2; theorem 3.1]

Dual of a Bailey pair is also a Bailey pair.

We shall make use of following known results in our analysis.

$${}_2\Phi_1\left[\begin{matrix} a^2, b; q; \dfrac{q^{1/2}}{b} \\ \dfrac{a^2 q}{b} \end{matrix}\right] = \frac{1}{2} \frac{(a^2, q^{1/2}; q)_\infty}{\left(\dfrac{a^2 q}{b}, \dfrac{q^{1/2}}{b}; q\right)_\infty} \times$$

$$\times \left\{ \frac{\left(\dfrac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(a; q^{1/2})_\infty} + \frac{\left(-\dfrac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(-a; q^{1/2})_\infty} \right\}, \qquad \left|\frac{q^{1/2}}{b}\right| < 1. \tag{1.18}$$

[4; p.74, Eq. (3.5)]

$$
{}_2\Phi_1\left[\begin{matrix} a^2, b; q; \dfrac{q^{3/2}}{b} \\ \dfrac{a^2q}{b} \end{matrix}\right] = \frac{1}{2a}\frac{(a^2, q^{1/2}; q)_\infty}{\left(\dfrac{a^2q}{b}, \dfrac{q^{1/2}}{b}; q\right)_\infty}\times
$$

$$
\times\left\{\frac{\left(\dfrac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(a; q^{1/2})_\infty} - \frac{\left(-\dfrac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(-a; q^{1/2})_\infty}\right\}, \qquad \left|\frac{q^{3/2}}{b}\right| < 1. \tag{1.19}
$$

[4; p.75, Eq. (3.6)]

$$
{}_4\Phi_3\left[\begin{matrix} a, c, \dfrac{aq^{\frac{1}{2}+n}}{c}, q^{-n}; q; q \\ \dfrac{aq}{c}, cq^{\frac{1}{2}-n}, aq^{1+n} \end{matrix}\right] = \frac{(a;q)_{n+1}(q^{1/2};q)_n\left(\dfrac{\sqrt{aq}}{c}; q^{1/2}\right)_{2n}}{2(aq/c;q)_n(q^{1/2}/c;q)_n(a^{1/2};q^{1/2})_{2n+1}}
$$

$$
+\frac{(a;q)_{n+1}(q^{1/2};q)_n\left(-\dfrac{\sqrt{aq}}{c}; q^{1/2}\right)_{2n}}{2(aq/c;q)_n(q^{1/2}/c;q)_n(-a^{1/2};q^{1/2})_{2n+1}} \tag{1.20}
$$

[4; p.71, Eq. (1.3)]

$$
{}_4\Phi_3\left[\begin{matrix} a, c, \dfrac{aq^{\frac{1}{2}+n}}{c}, q^{-n}; q; q^2 \\ \dfrac{aq}{c}, cq^{\frac{1}{2}-n}, aq^{1+n} \end{matrix}\right] = \frac{(a;q)_{n+1}(q^{1/2};q)_n\left(\dfrac{\sqrt{aq}}{c}; q^{1/2}\right)_{2n}}{2\sqrt{a}(aq/c;q)_n(q^{1/2}/c;q)_n(a^{1/2};q^{1/2})_{2n+1}}
$$

$$
-\frac{(a;q)_{n+1}(q^{1/2};q)_n\left(-\dfrac{\sqrt{aq}}{c}; q^{1/2}\right)_{2n}}{2\sqrt{a}(aq/c;q)_n(q^{1/2}/c;q)_n(-a^{1/2};q^{1/2})_{2n+1}} \tag{1.21}
$$

[4; p.77, Eq. (4.4)]

$$
\sum_{n=0}^{\infty}\sum_{r=0}^{n} A(n,r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A(n+r,r). \tag{1.22}
$$

[3; p.100, Eq. (2.1.2)]

2. Conjugate WP-Bailey pairs and related results

(i) If we choose $\delta_r(a,k,q) = \left(\frac{a}{k}q^{1/2}\right)^r$ in (1.14) and then using the summation formula (1.18) we get,

$$\gamma_n(a,k,q) = \frac{1}{2}\frac{(k,q^{1/2};q)_\infty}{(aq,aq^{1/2};q)_\infty}\left(\frac{a}{k}q^{1/2}\right)^n \times$$
$$\times\left\{\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty (k^{1/2}q^{1/2};q^{1/2})_n}{(k^{1/2}q^{1/2};q^{1/2})_\infty\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}+\frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty (-k^{1/2}q^{1/2};q^{1/2})_n}{(-k^{1/2}q^{1/2};q^{1/2})_\infty\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\right\}, \tag{2.1}$$

which yields the following result.

Theorem 1

For the WP-Bailey pair $\langle\alpha_n(a,k,q),\beta_n(a,k,q)\rangle$ satisfy (1.13) we have

$$\frac{1}{2}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(k^{1/2}q^{1/2};q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{(k^{1/2}q^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}q^{1/2}\right)^n\alpha_n(a,k,q)$$
$$+\frac{1}{2}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(-k^{1/2}q^{1/2};q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{(-k^{1/2}q^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}q^{1/2}\right)^n\alpha_n(a,k,q)$$
$$=\sum_{n=0}^{\infty}\left(\frac{a}{k}q^{1/2}\right)\beta_n(a,k,q), \tag{2.2}$$

provided the infinite series are all convergent.

(ii) If we choose $\delta_r(a,k,q) = \left(\frac{a}{k}q^{3/2}\right)^r$ in (1.14) and make use of the summation formula (1.19) we get,

$$\gamma_n(a,k,q) = \frac{1}{2k^{1/2}}\frac{(k,q^{1/2};q)_\infty}{(aq,aq^{1/2}/k;q)_\infty}\left(\frac{a}{k}q^{3/2}\right)^n \times$$
$$\times\left\{\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty (k^{1/2};q^{1/2})_n}{(k^{1/2};q^{1/2})_\infty\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}-\frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty (-k^{1/2};q^{1/2})_n}{(-k^{1/2};q^{1/2})_\infty\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\right\}. \tag{2.3}$$

The values of $\delta_n(a,k,q)$ and $\gamma_n(a,k,q)$ given in (2.3) together with (1.15) gives the following result

Theorem 2

If $\langle\alpha_n(a,k,q),\beta_n(a,k,q)\rangle$ satisfy (1.13) then under suitable convergence conditions we have,

$$\frac{1}{2k^{1/2}}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(k^{1/2};q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{(k^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}q^{1/2}\right)^n\alpha_n(a,k,q)$$

$$-\frac{1}{2k^{1/2}}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(-k^{1/2};q^{1/2})_\infty}\times$$

$$\times\sum_{n=0}^{\infty}\frac{(-k^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}q^{1/2}\right)^n\alpha_n(a,k,q)$$

$$=\sum_{n=0}^{\infty}\left(\frac{a}{k}q^{3/2}\right)\beta_n(a,k,q), \tag{2.4}$$

3. WP-Bailey pairs and dual WP-Bailey pairs

(i) If we set $c=\frac{a}{k}q^{1/2}$ in (1.20) we get,

$$\sum_{r=0}^{n}\frac{(a,aq^{1/2}/k,kq^n,q^{-n};q)_r q^r}{(q,kq^{1/2},aq^{1-n}/k,aq^{1+n};q)_r}$$

$$=\left(\frac{1+\sqrt{a}}{2}\right)\frac{(aq,\sqrt{q};q)_n}{(kq^{1/2},k/a;q)_n}\frac{(k/\sqrt{a},k\sqrt{q/a};q)_n}{(\sqrt{aq},q\sqrt{a};q)_n}$$

$$-\left(\frac{1-\sqrt{a}}{2}\right)\frac{(aq,\sqrt{q};q)_n}{(kq^{1/2},k/a;q)_n}\frac{(-k/\sqrt{a},-k\sqrt{q/a};q)_n}{(-\sqrt{aq},-q\sqrt{a};q)_n}. \tag{3.1}$$

Now, if we choose $\alpha_n(a,k,q)=\frac{(a,aq^{1/2}/k;q)_n}{(q,kq^{1/2};q)_n}\left(\frac{k}{a}\right)^n$ in (1.13) and making use of (3.1) we find,

$$\beta_n(a,k,q)=\left(\frac{1+\sqrt{a}}{2}\right)\frac{(k,\sqrt{q},k/\sqrt{a},k\sqrt{q/a};q)_n}{(q,k\sqrt{q},\sqrt{aq},q\sqrt{a};q)_n}$$

$$+\left(\frac{1-\sqrt{a}}{2}\right)\frac{(k,\sqrt{q},-k/\sqrt{a},-k\sqrt{q/a};q)_n}{(q,k\sqrt{q},-\sqrt{aq},-q\sqrt{a};q)_n}. \tag{3.2}$$

Thus, $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ given in (3.2) form a WP-Bailey pair.
Putting $c=\frac{a}{k}q^{1/2}$ in (1.21) we have,

$$\sum_{r=0}^{n}\frac{(a,aq^{1/2}/k,kq^n,q^{-n};q)_r q^{2r}}{(q,kq^{1/2},aq^{1-n}/k,aq^{1+n};q)_r}$$

$$\frac{1+\sqrt{a}}{2\sqrt{a}}\frac{(aq,\sqrt{q},k/\sqrt{a},k\sqrt{q/a};q)_n}{(kq^{1/2},kq^{1/2}/a,\sqrt{aq},q\sqrt{a};q)_n}-\frac{1-\sqrt{a}}{2\sqrt{a}}\frac{(aq,\sqrt{q},-k/\sqrt{a},-k\sqrt{q/a};q)_n}{(kq^{1/2},kq^{1/2}/a,-\sqrt{aq},-q\sqrt{a};q)_n} \tag{3.3}$$

Thus, if we choose $\alpha_n(a,k,q)=\frac{(a,aq^{1/2}/k;q)_n}{(q,kq^{1/2};q)_n}\left(\frac{kq}{a}\right)^n$ in (1.13) and make use of (3.3) we find,

$$\beta_n(a,k,q)=\frac{1+\sqrt{a}}{2\sqrt{a}}\frac{(k,\sqrt{q},k/\sqrt{a},k\sqrt{q/a};q)_n}{(q,kq^{1/2},\sqrt{aq},q\sqrt{a};q)_n}$$

$$-\frac{1-\sqrt{a}}{2\sqrt{a}}\frac{(k,\sqrt{q},-k/\sqrt{a},-k\sqrt{q/a};q)_n}{(q,kq^{1/2},-\sqrt{aq},-q\sqrt{a};q)_n}. \tag{3.4}$$

Here, $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ given in (3.4) are WP-Bailey pair. One remarkable observation is that WP-Bailey pairs given in (3.2) and (3.4) are dual of each other.

4. Transformation Formulae

(i) Substituting the WP-Bailey pair given in (3.2) in theorem 1, we obtain,

$$\frac{1}{2}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(k^{1/2}q^{1/2};q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{(k^{1/2}q^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(q^{1/2}\right)^n\frac{(a,aq^{1/2}/k;q)_n}{(q,kq^{1/2};q)_n}$$

$$+\frac{1}{2}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(-k^{1/2}q^{1/2};q^{1/2})_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(-k^{1/2}q^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(q^{1/2}\right)^n\frac{(a,aq^{1/2}/k;q)_n}{(q,kq^{1/2};q)_n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{k} q^{1/2}\right)^n \left\{ \frac{1+\sqrt{a}}{2} \frac{(k, \sqrt{q}, k/\sqrt{a}, k\sqrt{q/a}; q)_n}{(q, kq^{1/2}, \sqrt{aq}, q\sqrt{a}; q)_n} + \frac{1-\sqrt{a}}{2} \frac{(k, \sqrt{q}, -k/\sqrt{a}, -k\sqrt{q/a}; q)_n}{(q, kq^{1/2}, -\sqrt{aq}, -q\sqrt{a}; q)_n} \right\}, \tag{4.1}$$

where $\left|\frac{a}{k} q^{1/2}\right| < 1$.

(ii) Substituting the WP-Bailey pair given in (3.4) in theorem 1, we obtain,

$$\frac{1}{2} \frac{(k, q^{1/2}; q)_\infty}{\left(aq, \frac{aq^{1/2}}{k}; q\right)_\infty} \frac{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(k^{1/2}q^{1/2}; q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(k^{1/2}q^{1/2}; q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n} \left(q^{3/2}\right)^n \frac{(a, aq^{1/2}/k; q)_n}{(q, kq^{1/2}; q)_n}$$

$$+ \frac{1}{2} \frac{(k, q^{1/2}; q)_\infty}{\left(aq, \frac{aq^{1/2}}{k}; q\right)_\infty} \frac{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(-k^{1/2}q^{1/2}; q^{1/2})_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(-k^{1/2}q^{1/2}; q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n} \left(q^{3/2}\right)^n \frac{(a, aq^{1/2}/k; q)_n}{(q, kq^{1/2}; q)_n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{k} q^{1/2}\right)^n \left\{ \frac{1+\sqrt{a}}{2\sqrt{a}} \frac{(k, \sqrt{q}, k/\sqrt{a}, k\sqrt{q/a}; q)_n}{(q, kq^{1/2}, \sqrt{aq}, q\sqrt{a}; q)_n} - \frac{1-\sqrt{a}}{2\sqrt{a}} \frac{(k, \sqrt{q}, -k/\sqrt{a}, -k\sqrt{q/a}; q)_n}{(q, kq^{1/2}, -\sqrt{aq}, -q\sqrt{a}; q)_n} \right\}. \tag{4.2}$$

(iii) Putting the WP-Bailey pair given in (3.2) in theorem 2 we obtain,

$$\frac{1}{2\sqrt{k}} \frac{(k, q^{1/2}; q)_\infty}{\left(aq, \frac{aq^{1/2}}{k}; q\right)_\infty} \frac{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(k^{1/2}; q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(k^{1/2}; q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n} \frac{(a, aq^{1/2}/k; q)_n}{(q, kq^{1/2}; q)_n} \left(q^{n/2}\right)$$

$$- \frac{1}{2\sqrt{k}} \frac{(k, q^{1/2}; q)_\infty}{\left(aq, \frac{aq^{1/2}}{k}; q\right)_\infty} \frac{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(-k^{1/2}; q^{1/2})_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(-k^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n} \frac{(a,aq^{1/2}/k;q)_n}{(q,kq^{1/2};q)_n} \left(q^{n/2}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{k}q^{3/2}\right)^n \left\{ \frac{1+\sqrt{a}}{2} \frac{(k,\sqrt{q},k/\sqrt{a},k\sqrt{q/a};q)_n}{(q,kq^{1/2},\sqrt{aq},q\sqrt{a};q)_n} + \frac{1-\sqrt{a}}{2} \frac{(k,\sqrt{q},-k/\sqrt{a},-k\sqrt{q/a};q)_n}{(q,kq^{1/2},-\sqrt{aq},-q\sqrt{a};q)_n} \right\}. \tag{4.3}$$

(iv) Putting the WP-Bailey pair given in (3.4) in theorem 2 we obtain,

$$\frac{1}{2\sqrt{k}} \frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty} \frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(k^{1/2};q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(k^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n} \frac{(a,aq^{1/2}/k;q)_n}{(q,kq^{1/2};q)_n} \left(q^{3n/2}\right)$$

$$-\frac{1}{2\sqrt{k}} \frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty} \frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(-k^{1/2};q^{1/2})_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(-k^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n} \frac{(a,aq^{1/2}/k;q)_n}{(q,kq^{1/2};q)_n} \left(q^{3n/2}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{k}q^{3/2}\right)^n \left\{ \frac{1+\sqrt{a}}{2\sqrt{a}} \frac{(k,\sqrt{q},k/\sqrt{a},k\sqrt{q/a};q)_n}{(q,kq^{1/2},\sqrt{aq},q\sqrt{a};q)_n} - \frac{1-\sqrt{a}}{2\sqrt{a}} \frac{(k,\sqrt{q},-k/\sqrt{a},-k\sqrt{q/a};q)_n}{(q,kq^{1/2},-\sqrt{aq},-q\sqrt{a};q)_n} \right\}. \tag{4.4}$$

5. Some more useful theorems

Theorem 3. If we put the value of $\beta_n(a,k,q)$ from (1.13) in (2.2) then after using the identity (1.22) we get,

$$\frac{1}{2} \frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty} \frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(k^{1/2}q^{1/2};q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(k^{1/2}q^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n} \left(\frac{a}{k}q^{1/2}\right)^n \alpha_n(a,k,q)$$

$$+\frac{1}{2}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(-k^{1/2}q^{1/2};q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{(-k^{1/2}q^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}q^{1/2}\right)^n\alpha_n(a,k,q)$$

$$=\sum_{n,r=0}^{\infty}\left(\frac{a}{k}q^{1/2}\right)^{n+r}\frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}\alpha_r(a,k,q), \quad (5.1)$$

where $\alpha_n(a,k,q)$ is an arbitrary sequence such that $\alpha_0(a,k,q)=1$.

Theorem 4. If we put the value of $\beta_n(a,k,q)$ from (1.13) in (2.3) then apply the identity (1.22) we get,

$$\frac{1}{2\sqrt{k}}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(k^{1/2};q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{(k^{1/2};q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}q^{1/2}\right)^n\alpha_n(a,k,q)$$

$$-\frac{1}{2\sqrt{k}}\frac{(k,q^{1/2};q)_\infty}{\left(aq,\frac{aq^{1/2}}{k};q\right)_\infty}\frac{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_\infty}{(-k^{1/2};q^{1/2})_\infty}\sum_{n=0}^{\infty}\frac{(-k^{1/2};q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}};q^{1/2}\right)_n}\left(\frac{a}{k}q^{1/2}\right)^n\alpha_n(a,k,q)$$

$$=\sum_{n,r=0}^{\infty}\left(\frac{a}{k}q^{3/2}\right)^{n+r}\frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}\alpha_r(a,k,q), \quad (5.2)$$

where $\alpha_n(a,k,q)$ is an arbitrary sequence such that $\alpha_0(a,k,q)=1$.
From (5.1) and (5.2) one can easily establish double series identities.

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