South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 1 (2025), pp. 201-216

DOI: 10.56827/SEAJMMS.2025.2101.16 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

THE R-MATRIX COMPLETION PROBLEM

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(**Received:** *May 19, 2024* **Accepted:** *Apr. 23, 2025* **Published:** *Apr. 30, 2025)*

Abstract: Matrix completion problem (MCP) is a very well-established process of rebuilding a matrix's unknown elements. A $m \times m$ matrix B is a R-matrix if for every $r = 1, 2, \ldots, m$, the sum of all $r \times r$ principal minors of B is negative. A digraph D possesses R-completion if it is possible to complete any partial R-matrix that defines D to a R-matrix. In this article we have examined the R-matrix completion problem. Here some necessary as well as some sufficient conditions for a digraph to have the R-completion are discussed. In addition, the digraphs of order up to four that possesses R-completion have been categorized. Finally, a comparative discussion between R-matrix completion and N-matrix completion is provided.

Keywords and Phrases: Partial matrix, R-matrix, Matrix completion, Digraph.

2020 Mathematics Subject Classification: 15A48.

1. Introduction

In Matrix completion problem the missing elements of a partially observed matrix are estimated or filled in. This problem arises in various fields such as machine learning, statistics, signal processing and different allied areas. The goal of MCP is to predict the missing values in the matrix based on the observed entries and certain assumptions about the underlying structure of the data.

MCPs are studied for numerous varieties of matrices, but the utmost significant class of matrices is positive definite (PD) matrices. In 1984 Burg initiated matrix completion problems by studying the positive definite completion [4]. Dym and Gohberg considered the completion problem of PD tri-diagonal matrices [10]. Finally Grone et al answered the whole question for PD matrices in 1984 [12] and derived that only chordal graphs possesses PD Completion. Over the most recent twenty years numerous analysts concentrated on various MCPs (e.g., [6, 7, 9, 11, 14, 15, 16, 17, 18, 19, 20, 23, 24, 25, 26, 27]) and applied them to in various domains. The basic idea of MCP is to reconstruct a real $m \times m$ partial matrix from a subset of its entries, where some entries are missing or unknown to a desired type of matrix in combinatorial approach.

A real $m \times m$ matrix $A = [a_{ij}]$ is a *N-matrix* (N_0-matrix) if for every $r = 1, 2, \ldots, m$, all $r \times r$ principal minors of A are negative(non-positive). A real $m \times m$ matrix $B = [b_{ij}]$ is a *R-matrix* (R_0-matrix) if for every $r = 1, 2, \ldots, m$, $S_r(B) < 0$ ($S_r(B) \leq 0$), where $S_r(B)$ refers to the sum of all principal minors of size $r \times r$. Clearly every N-matrix is a R-matrix but converse is not true. This motivates us to study R-matrix completion problem in a illustrative manner. The completion problem for N-matrix and its associated classes were discussed in [1, 2, 3]. Further, a *partial matrix* refers to a matrix where certain entries are specified while others remain unspecified. A process which completes a partial matrix is known as *completion*, wherein values are assigned to the unspecified entries to achieve desired type of matrix. A partial matrix C_1 is said to be a partial N-matrix if every completely specified principal sub-matrix of C_1 is an N-matrix. In a parallel way, a partial matrix C_2 is a partial R-matrix if $S_r(C_2) < 0$ for every $r = 1, 2, \ldots, m$ for which all $r \times r$ principal submatrices are fully specified.

Graphs and digraphs have a strong relationship to the MCPs. The graph theoretic principles utilized here can be located in any conventional publication i.e. [5, 13]. Nevertheless, we kindly ask the reader to consult any of the sources [21, 22, 23] for initial terms and definitions utilized in this article.

Partial R-matrix and the R-matrix completion problem

For any MCP the role of a partial matrix is very important. So a proper characterization of partial matrices is very much essential to study MCP of a specific class of matrices. The following proposition provides an ideal depiction of a partial R-matrix:

Proposition 1.1. *A partial matrix $N = [n_{ij}]$ is a partial R-matrix iff precisely one of the subsequent events occurs:*

(i) *One diagonal entry of matrix N is omitted.*

(ii) *N has diagonals that have been specified and trace$(N) < 0$. N omits an off* *diagonal entry.*

(iii) *N is complete R-matrix.*

If there exists a partial R-matrix N that can be extended to form a complete R-matrix M, then M is called the R-completion of N. A digraph $D = (V_D, A_D)$ possesses a *R-completion* if it is possible to complete any partial R-matrix that specifies D into a full R-matrix. The primary objective of studying R-matrix completion is to categorize all the digraphs based on R-matrix completion.

It can be confirmed that if a partial R-matrix N possesses R-matrix completion, denoted as M, then any partial R-matrix under permutation similarity to N also possesses R-matrix completion. Similarly, if a digraph D possesses a R-matrix completion, then any digraph that is isomorphic to D will possess a R-completion. Firstly we think about a partial R-matrix N_1 with empty diagonals. Clearly N_1 has R-matrix completion. In-fact assigning a significantly negative value to a diagonal elements and considering the rest of all diagonal entries as 1 of N_1 we will get our desired R-completion. Now,, we consider a partial R-matrix N_2 with some specified diagonal. Additionally, we presume that the diagonals at $(k, k), (k = i + 1, \ldots, n)$ places of N_2 are unspecified. Even if $N_2(1, 2, \ldots, k)$ is complete, N_2 fails to attain R-completion. For instance, take the partial R-matrix

$$N_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & * \end{bmatrix},$$

with $*$ as an unspecified entry. For any completion of N_2 we always have $\det(N_2) = 0$. Therefore, the attempt to extend N_2 into an R-matrix becomes unsuccessful. Again $N_2(1, 2, 3, \ldots, k)$ with an unspecified entry has a R-completion, then N_2 has R-completion. Based on the above discussion both the Remark 1.2 and Remark 1.3 are obtained:

Remark 1.2. *A digraph which excludes all loops has R-completion.*

Remark 1.3. *Suppose D be a digraph and $\gamma \subseteq V_D$ at which D contains loops. If $D_\gamma \neq K_{|\gamma|}$, the induced sub-digraph of D has R-completion, then D possesses R-completion.*

For any β class of matrices say P, P_0, (w)ss P, positive Q, non-negative Q, Q_0, (w)ss Q-matrices [8, 18, 21, 22, 23], it has been observed that the β completion of every component (strong component) in a directed graph D implies the β-completion of the entire digraph D. The reverse of the result does not holds for Q-matrices and its sub-classes as well as Q_0-matrices although the result holds for P-matrices and its sub-classes. However this particular result fails in case of R-matrix completion. For instance, consider the partial R-matrix

$$N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & x & 0 \\ 0 & y & -1 \end{bmatrix},$$

where x and y are not specified. In this context, N_3 represents a digraph D where the strongly connected components $D_{\{1,2\}}$ and $D_{\{3\}}$ possess R-completion. In spite of that D fails to attain R-completion since $\det N_3 = 1 > 0$ for any x, y.

In similar way, if a digraph D_1 possesses R-completion, it does not necessarily imply that induced sub digraphs of D_1 will also have R-completion. This characteristic of R-completion problem distinguishes the R-completion from the MCP of P-matrix and its sub-classes.

Example 1.4. Consider the partial R-matrix

$$N_4 = \begin{bmatrix} n_{11} & n_{12} & * \\ n_{21} & * & n_{23} \\ * & n_{32} & * \end{bmatrix},$$

with $*$ as the unspecified entries. Consider $M_4(s)$, a completion of of N_4 described as follows:

$$M_4(s) = \begin{bmatrix} n_{11} & n_{12} & s \\ n_{21} & s & n_{23} \\ s & n_{32} & -2s \end{bmatrix}.$$

Then for $1 \leq j \leq 3$, we have

$$S_j(M_4(s)) = -c_j s^j + p(s) < 0,$$

where c_j is a positive constant and $p(s)$ is a polynomial in s such that $deg(p(s)) \leq j-1$. For sufficiently large s, we have $S_j(M_4(s)) < 0, 1 \leq j \leq 3$. Accordingly, M_4 has a R-completion. Furthermore, we think about the submatrix $M_4[1,2]$ induced by the diagonals $\{1,2\}$ of M_4 as following:

$$M_4[1,2] = \begin{bmatrix} 0 & 0 \\ 1 & * \end{bmatrix}.$$

For any value of $*$, $M_4[1,2]$ fails to attain a R-completion. Thus, the principal sub matrix of any partial R-matrix fails to inherit R-completion.

2. Sufficient conditions for R-completion

Theorem 2.1. *Any spanning sub-digraph of a digraph $D \neq K_m^*$ with R-completion also possesses R-completion.*

Proof. Take $\tilde{D}$, a spanning sub-digraph of D and $\tilde{N}$ is the partial R-matrix identifying $\tilde{D}$. Now, we can easily obtain a partial R-matrix N specifying D from partial R-matrix $\tilde{N}$ by choosing the entries corresponding to $(i,j) \in A_D \setminus A_{\tilde{D}}$ as 0. As a result, N becomes a partial R-matrix specifying D. As D has R-completion, a R-completion M of N can easily be obtained. Clearly M is also a R-completion of $\tilde{N}$.

Theorem 2.2. *Consider a digraph $H \neq K_m^*$. If a weight function Ω can be defined on $\overline{H}$ such that for each $r \in \{1,2,\ldots,m\}$, there exists a negative permutation digraph η_r of order r in $\overline{H}$ satisfying,*

(i) $\Omega^*(\eta_r) > \Omega^*(\eta)$ *for every permutation subdigraph of positive weight η of H of size r and*

(ii) $\Omega^*(\eta_r) > \sum_{e \in H_1} \Omega(e)$ *where $H_1 \subseteq A_H, |H_1| \leq r-1$,*

then H has R-completion.

Proof. Consider the partial R-matrix $N = [n_{lk}]$, which identifies H. For $t > 1$, a completion $M(t) = [m_{lk}]$ of N can be obtained as follows:

$$m_{lk} = \begin{cases} n_{lk}, & \text{if } (l,k) \in A_H \\ t^{\Omega(e)}, & \text{if } (l,k) \in A_{\overline{H}} \end{cases}$$

Now,, $\forall\ r \in \{1,2,\ldots,m\}$ any negative permutation digraph η_r contains a term $t^{\Omega^*(\eta_r)}$ with a negative coefficient in $S_r(M(t))$. Although any positive permutation digraph η is either a positive permutation subdigraph of H of order r or a permutation sub-digraph of K_m comprising a maximum of $r-1$ arcs from H. Upon thorough examination of Ω, any positive permutation digraph η adds a factor of δt^s, where $\delta \geq 0$ and $s < \Omega^*(\eta_r)$ in $S_r(M(t))$. By opting vast values of t we have $S_r(M(t)) < 0$ for $r = 1,\ldots,m$ and as a consequence H has R-completion.

Our next example demonstrates that the existence of R-completion can be shown using Theorem 2.2.

Example 2.3. Suppose

$$N_5 = \begin{bmatrix} n_{11} & * & * & n_{14} \\ n_{21} & n_{22} & n_{23} & * \\ * & n_{32} & n_{33} & n_{34} \\ * & * & n_{43} & n_{44} \end{bmatrix},$$

be a partial R-matrix identifying D_2 in Figure 1.

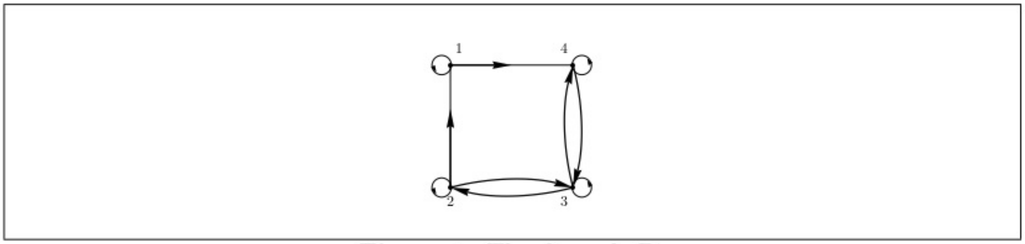

Figure 1: The digraph D_2

The digraph D_2 has R-completion by Theorem 2.2. For $s > 0$, a completion $M_5(s, \mu) = [m_{ij}]$ of N_5 can be acquired as follows:

$$M_5(s) = \begin{bmatrix} n_{11} & s^3 & s^3 & n_{14} \\ n_{21} & n_{22} & n_{23} & -s \\ s^3 & n_{32} & n_{33} & n_{34} \\ s^3 & s & n_{43} & n_{44} \end{bmatrix},$$

Now,

$$\begin{aligned} S_1(M_5) &= \sum_{i=1}^{4} n_{ii} \quad , \\ S_2(M_5) &= \quad -s^6 + p_1(s), \\ S_3(A_5) &= \quad -s^7 + p_2(s), \\ S_4(A_5) &= \quad -s^8 + p_3(s), \end{aligned}$$

where $p_j(s), j = 1, 2, 3$ are polynomials in s of total degree less than $6, 7, 8$ respectively. Choosing extremely large value of s, we have $S_t(M_5) < 0;\ \forall \quad t = 1, 2, 3, 4$ and M_5 is the desired R-completion of N_5.

Remark 2.4. *The converse of Theorem 2.2 is still not established for the R-completion in general. It has come to our attention that the converse holds true for the digraphs with an order of up to 5. But overall the effectiveness of converse of Theorem 2.2 is still unsolved.*

Theorem 2.5. *Suppose $\overline{D}$, the complement of a digraph $D \neq K_4^*$ is stratified and $\overline{D}$ contains only one 2-cycle. If it is possible to sign the arcs of $\overline{D}$ in such a way that the sign of every cycle in $\overline{D}$ becomes negative, then D has R-completion.*

Proof. Consider a partial R-matrix $N = [n_{ij}]$ which specifies D. For any $s > 0$ a completion M of N can be obtained by assigning unspecified entry $n_{ij} = sgn(i, j)s$

(with the help of signing of arcs of $\overline{D}$). Now,, for each $r = 2, 3, 4$ we have $S_r(M) = -c_r s^r + p_r(t)$, where c_r indicates total count of r order permutation subdigraphs in D and $p_r(s)$ is a polynomial such that $\deg(p_r(s)) < r$. Now,, if D includes all loops, then the trace of M turns out negative. Now,, by opting s sufficiently large, M turns out a R-matrix.

A fan $F_m = \langle V(F_m), A(F_m)\rangle$ is a digraph of order m such that $V(F_m) = \langle m\rangle, A(F_m) = P_m \cup S_m$, where P_m is the path $1 \to 2 \cdots \to m$ and S_m is the star with arcs $(r, 1)$, $r = 2, \ldots, m$. In our next result we will see that F_m plays an important role in R-completion.

Theorem 2.6. *Suppose $D \neq K_m^*$ be a digraph such that F_m is a subdigraph of $\overline{D}$, then D has R-completion.*

Proof. Here by definition F_m is stratified. Now,, we sign the arcs of F_m in the following way: $\operatorname{sgn}(r, r+1) = +$, $r = 1, 2, 3, \ldots, m$ and $\operatorname{sgn}(r, 1) = -$ if r is odd and $\operatorname{sgn}(r, 1) = +$ if r is even. In that case the sign of every cycle becomes negative and clearly F_m has R-completion.

Now, we will talk about some necessary conditions for R-completion in upcoming section.

3. Necessary conditions for R-completion

Theorem 3.1. *Suppose $D \neq K_m^*, m \geq 2$, forbids at least one loop. If D has R-completion, then $\overline{D}$ must be stratified.*

Proof. Consider $r \geq 2$ and $\overline{D}$ does not have a r-order permutation digraph. Suppose the partial matrix N specifies the digraph D. Then for any completion M of N, we have $S_r(M) = 0$ and M is not a R-matrix. Hence stratification of $\overline{D}$ is essential.

However the converse of the Theorem 3.1 does not hold. The next example illustrates this.

Example 3.2. In Figure 2 we consider a digraph D_3. Here $\overline{D_3}$ is stratified. Although D_3 fails to attain R-completion.

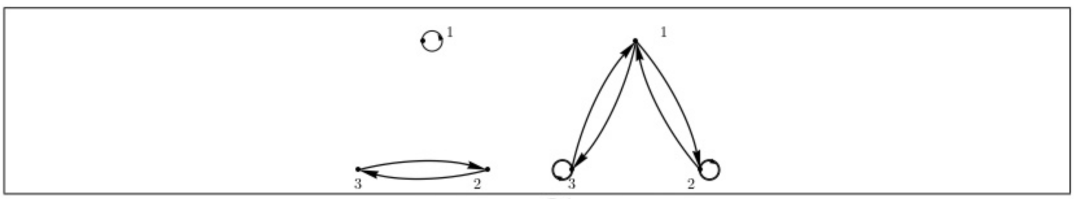

Figure 2: $\widetilde{D_3}$ and D_3

Suppose N_6 where

$$N_6 = \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & z & 0 \end{bmatrix},$$

a partial R-matrix specifying D_3 with unspecified entries as x, y, z. Now, we have $S_1(N_6) = x < 0$ and $S_2(N_6) = -yz < 0$. Finally $\det(N_6) = -xyz$, which cannot be negative since $x, -yz$ are both negative. Thus N_6 fails to attain R-matrix completion for any choice of unspecified entries x, y, z.

Corollary 3.3. *Suppose $D \neq K_m^*$ be a digraph that forbids at least one loop and $|A_D| > m(m-1)$. Then D fails to have R-completion.*

Proof. Here $\overline{D}$ contains lesser than $m^2 - m(m-1) = m$ arcs and consequently $\overline{D}$ cannot be stratified. Hence the result follows.

Now, we consider the matter when a digraph D includes all loops. In this case stratification of $\overline{D}$ is not essential. Our next theorem illustrates that for R-completion of a digraph D, $\overline{D}$ cannot be asymmetric.

Theorem 3.4. *If a digraph $D \neq K_m^*, m \geq 2$, with all loops has R-completion, then $\overline{D}$ cannot be asymmetric.*

Proof. Suppose $\overline{D}$ is asymmetric. Consider the partial R-matrix $N = [n_{ij}]$ specifying D as follows:

$$n_{ij} = \begin{cases} -1, & \text{if } (i,j) = (1,1) \in A_D \\ x_{ij}, & \text{if } (i,j) \in A_{\overline{D}} \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any R-completion M of N, $S_2(M) = 0$. Hence, a contradiction arises. Thus $\overline{D}$ cannot be asymmetric.

The converse of the Theorem 3.4 is not valid. Our next example illustrates this.

Example 3.5. Take a digraph D_4 in Figure 3. Here $\overline{D_4}$ is symmetric, however D_4 does not have R-completion.

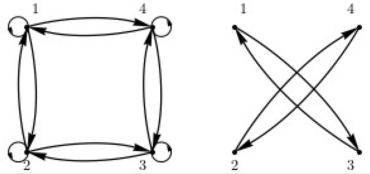

Figure 3: The digraph D_4 and $\overline{D_4}$

Consider

$$N_7 = \begin{bmatrix} -1 & 0 & x & 0 \\ 0 & -1 & 0 & z \\ y & 0 & 0 & 0 \\ 0 & w & 0 & 0 \end{bmatrix},$$

a partial R-matrix identifying the digraph D_4 with unspecified entries as x, y, z, w. Now, for any R-completion of N_7, we have $S_2(N_7) = 1 - xy - zw < 0$ and $S_3(N_7) = xy + zw < 0$. Clearly the conditions in $S_2(N_7)$ and $S_3(N_7)$ contradict each other. Hence N_7 fails to attain R-completion.

Our next corollary can be directly obtained from the Theorem 3.4.

Corollary 3.6. *If a digraph $D \neq K_m^*$ contains all loops and has R-completion, then D must not either a tournament or subdigraph of a tournament.*

We have seen that for R-completion of a digraph D containing all loops, $\overline{D}$ cannot be asymmetric. Also we have noticed that a digraph whose complement is symmetric may not have R-completion. So to have R-completion weak stratification is also needed for a digraph with all loops.

Theorem 3.7. *If a digraph $D \neq K_m^*$ with all loops has R-completion, then D must be weakly stratified.*

Proof. Let $r \geq 2$ and neither $\overline{D}$ contains a r order permutation digraph nor for any vertex $v \in V_{\overline{D}}$, $\overline{D} - v$ does not contain a $r-1$ order permutation digraph. Let $N = [n_{lk}]$ be a partial matrix identifying D with $n_{vv} = -1$ and all other specified entries as 0. Then for any completion M of N we have $S_r(M) = 0$. Hence R completion of M is not a possible.

Again the converse of the Theorem 3.7 does not hold. The digraph D_4 given in Example 3.5 is weakly stratified but it does not have R-completion.

Corollary 3.8. *Suppose $D \neq K_m^*$, a digraph with all loops has greater than $(m-1)^2 - 1$ non-loop arcs. Then D fails to have R-completion.*

We omit the proof since it is quite easy and straight forward.

4. Categorization of digraphs of order up-to 4 based on R-completion

In this segment, we address the outcomes from the previous segments in order to classify the digraphs with a maximum order of 4 which includes all loops as R-completion. The digraphs discussed in the following sections are named according to their sequence as mentioned in [[13], page. 233]. We denote $D_p(q, n)$ as the n-th digraph with p vertices and q (non-loop) arcs. For complete classification we have split this section in to series of lemmas.

Lemma 4.1. *The digraphs $D_4(q, n)$ given in tabular form fails to attain R-*

completion.

p	q	n
4	6	3– 6,8
4	7	2
4	8	2

Proof. In Example 3.5, we have seen that the digraph $D_2(4,8)$ fails to have R-completion.

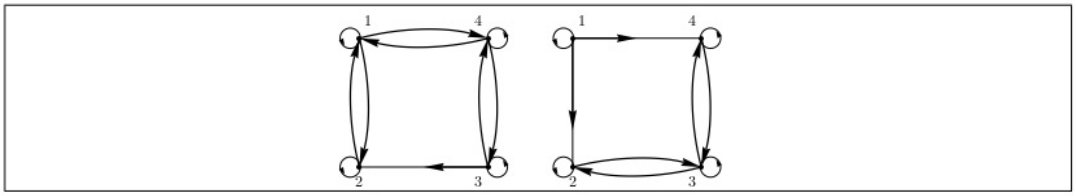

Figure 4: The digraph $D_2(4,7)$ and $D_8(4,6)$

Now, we consider

$$N_8 = \begin{bmatrix} -1 & 0 & x & 0 \\ q & -1 & 0 & z \\ y & 0 & 0 & 0 \\ u & w & 0 & 0 \end{bmatrix},$$

a partial R-matrix identifying $D_8(4,6)$ with unspecified entries as x, y, z, w, u, q. Now, for any R-completion of N_8, we have $S_2(N_8) = 1 - xy - zw < 0$ and $S_3(N_8) = xy + zw < 0$. Clearly the conditions in $S_2(N_8)$ and $S_3(N_8)$ contradict each other. Thus R-matrix completion of N_8 cannot be possible for any selection of unspecified entries x, y, z, w, u, q. Now, if we specify the entry in $(4,1)$-th entry of N_8 as 0, then it specifies a digraph which is isomorphic to $D_2(4,7)$. In the same way we can show that $D_2(4,7)$ does not possesses R-completion.

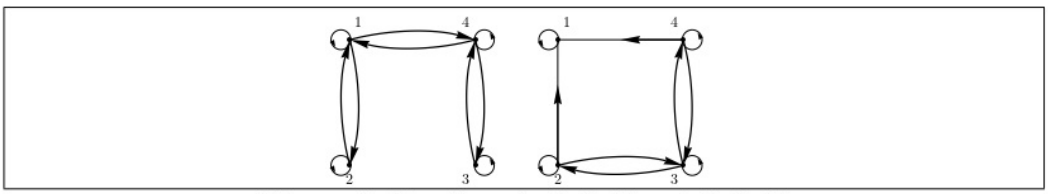

Figure 5: The digraph $D_3(4,6)$ and $D_6(4,6)$

Next we take a partial R-matrix

$$N_9 = \begin{bmatrix} -1 & 0 & x & 0 \\ 0 & -1 & u_1 & z \\ y & u_2 & 0 & 0 \\ 0 & w & 0 & 0 \end{bmatrix},$$

identifying $D_3(4,6)$ with unspecified entries as x, y, z, w, u_1, u_2. Now, for any R-completion of N_9, one can easily get two contradictory conditions from $S_2(N_9)$ and $S_3(N_9)$ which leads that N_9 cannot be completed to a R-matrix for any choice of x, y, z, w, u_1, u_2. Also any partial R-matrix specifying the digraph $D_6(4,6)$ is exactly the transpose of the matrix N_8 and hence $D_6(4,6)$ does not have R-completion.

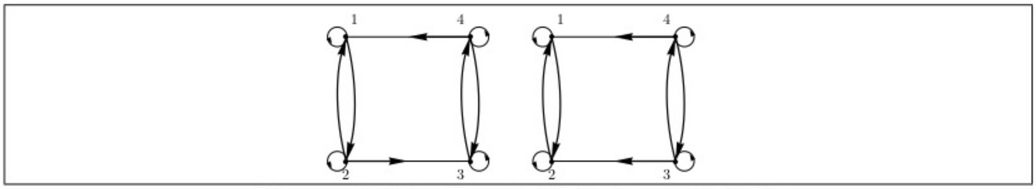

Figure 6: The digraph $D_4(4,6)$ and $D_5(4,6)$

Again we consider a partial R-matrix

$$N_{10} = \begin{bmatrix} -1 & 0 & x & 0 \\ 0 & -1 & 0 & z \\ y & x_1 & 0 & 0 \\ x_2 & w & 0 & 0 \end{bmatrix},$$

specifying the digraph $D_4(4,6)$ with unspecified entries as x, y, z, w, x_1, x_2. Now, for any R-completion of N_{10}, two contradictory conditions from $S_2(N_{10})$ and $S_3(N_{10})$ can be easily obtained and as a result R-completion of N_{10} cannot be possible for any choice of x, y, z, w, x_1, x_2. In a similar way one can see that $D_5(4,6)$ also does not have R-completion. As the proof is straightforward, we have omitted it.

Lemma 4.2. *The digraphs $D_p(q,n)$ given in tabular form fails to attain R-completion.*

p	q	n
2	1	1
3	2	1,3,4
3	3	1–4
3	4	1–4
3	5	1
4	3	8,11
4	4	10,12,14,15,21,27
4	5	4–6, 11,14–17,19,21–24,26,28,29,31,34,36,37
4	6	1,2,9–14, 15–23, ,26,27,29,30,32– 41,43– 48
4	7	1,3–38
4	8	1,3–27
4	9	1,3–27
4	10	1–5
4	11	1

Proof. Each of the digraphs listed above is neither stratified nor weakly stratified, so they do not have R completion.

Theorem 4.3. *The digraphs $D_p(q,n)$ given in tabular form have R-completion.*

p	q	n
1	0	1
2	0,2	1
3	0,6	1
3	1	1
3	2	3
4	0,12	1
4	1	1
4	2	1–5
4	3	1–7,9,10,12,13
4	4	1– 9,11,13,16–20,22–26
4	5	1–3,7–10,12,13,18,20,25,27,30,32,33,35,38
4	6	7,24,25,28,31,42

Proof. Since $D_1(0,1); D_2(0,1), D_2(2,1); D_3(0,1), D_3(6,1), D_4(0,1), D_4(12,1)$ is complete, by definition they have R-completion. Each of the digraph $D_3(q,n) : p = 3, q = 1,2, n = 1; D_4(q,n) : p = 4, q = 6, n = 7; q = 5; n = 2,9$ satisfies the Theorem 2.2, hence they have R-completion. Again, the complement $\overline{D_4(q,n)}$ of the digraph $D_4(q,n) : q = 6, n = 24, 25, 28, 31, 42; q = 5, n = 30, 32, 33, 35, 38$ is stratified and contains only one 2-cycle and one can easily sign the arcs of $\overline{D_4(q,n)}$ such that all cycles in $\overline{D_4(q,n)}$ become negative; thus, by Theorem 2.5 each digraph $D_4(q,n) : q = 6, n = 24, 25, 28, 31, 42; q = 5, n = 30, 32, 33, 35, 38$ has R-completion. Finally, the rest of any digraph is a spanning subdigraph of any one of the subdigraphs of $D_4(q,n) : p = 4, q = 6, n = 7; q = 5, n = 2, 9$, hence by Theorem 2.1 they have R-completion.

5. Comparison with N matrix completion

While every N matrix is indeed an R-matrix, the MCPs for these two classes exhibit stark differences. The following two cases demonstrate this.

1. The digraph D_5 in Figure 7 fails to attain N-completion (See [1]) but it has R-completion (See Theorem 4.3) .

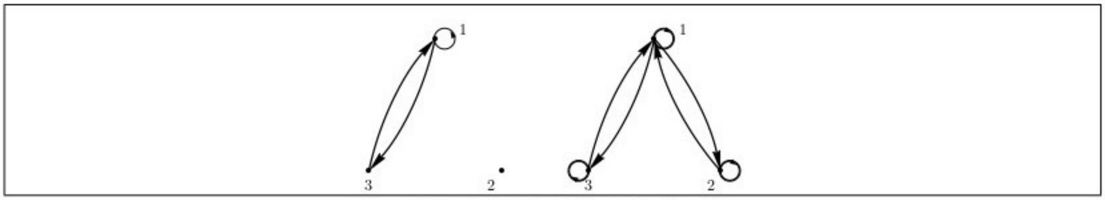

Figure 7: D_5 and D_6

2. On contrary consider the digraph D_6 in Figure 7. The digraph D_6 have N-completion (See [1]) but it does not have R-completion (See Lemma 4.2) .

6. Conclusion

This article has addressed the completion problem concerning the class of R-matrices. We have acquired the necessary and sufficient conditions for a directed graph to possess an R completion. We observed that a digraph containing all loops is incapable of achieving R-completion, unless it is a weakly stratified. Stratification is another crucial requirement for a digraph which omits a loop to have R-completion. The findings presented in this article provide us with a reasonable understanding of the R-completion of digraphs with low order. Also a comparison with N-matrix completion problem with our R-martrix completion problem is studied. It is seen that none of the completion problem implies each other. Still, the R-matrix completion is not yet fully understood, so it is not better to draw any kind of conclusion regarding relationship between two completion problems. Also the converse of the Theorem 2.2 is still open for all. Hence, one can involve deeper into the study of a digraph by seeking a complete characterization that has not yet been discovered, resulting in an R-completion.

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