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A STUDY OF A NOVEL CATEGORY OF MACROBERT-STYLE INTEGRALS INCORPORATING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract: In 2018, Masjed-Jamei and Koepf established interesting and valuable generalizations of various classical summation formulas for the generalized hypergeometric series ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_6F_5$. Building on this work, in this study, we establish seven generalized hypergeometric integrals of the MacRobert-style using these summation theorems. In addition to that, we present several special cases to illustrate the applicability of our results in the literature, including the most recent contributions by Kulkarni et al.

Keywords and Phrases: Generalized Hypergeometric function, Summation theorem, MacRobert integral.

2020 Mathematics Subject Classification: 33C05, 33C20.

1. Introduction

For any complex number $u \in \mathbb{C}$, the Pochhammer symbol or ascending factorial, introduced by Leo August Pochhammer [1, 25], is defined by

$$(u)_n = \begin{cases} 1, & (n = 0, \quad u \neq 0) \\ \prod_{r=0}^{n-1}(u+r), & (n \in \mathbb{N}) \end{cases} \\ = \frac{\Gamma(u+n)}{\Gamma(u)} \tag{1.1}$$

where $\Gamma(u)$ belongs to the category of special transcendental functions, It extends the concept of factorial to complex numbers and is widely recognized as the Gamma function. We characterize the expression through the integral given by

$$\Gamma(u) = \int_0^{\infty} t^{u-1} e^{-t}\, dt \quad \text{for } \mathrm{Re}(u) > 0.$$

Using the ascending factorial (1.1), the generalized hypergeometric function is expressed as [2, 4, 23, 25],

$${}_pF_q\left[\begin{array}{ccc} \rho_1, & \cdots & , \rho_p \\ \beta_1, & \cdots & , \beta_q \end{array} \mid z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(\rho_i)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!} \tag{1.2}$$

where $\beta_j \neq 0, -1, -2, \ldots$.

The parameters $\rho_1, \ldots, \rho_p \in \mathbb{C}$ represent the p parameters in the numerator, while $\beta_1, \ldots, \beta_q \in \mathbb{C}$ represent the q parameters in the denominator. For details regarding the convergence conditions of ${}_pF_q$, readers can refer to the standard textbooks [2, 3, 4].

When $p = 2$ and $q = 1$, the above series (1.2) takes the form

$${}_2F_1\left[\begin{array}{cc} \rho_1, & \rho_2 \\ & \beta_1 \end{array} \mid z\right] = \sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n}{(\beta)_n} \frac{z^n}{n!}$$

which converges for $|z| \leq 1$, and commonly known as Gauss's hypergeometric function.

When $p = 1, q = 1$, the series (1.2) takes the form:

$${}_1F_1\left[\begin{array}{c} \rho_1 \\ \beta_1 \end{array} \mid z\right] = \sum_{n=0}^{\infty} \frac{(\rho_1)_n}{(\beta_1)_n} \frac{z^n}{n!}$$

Which converges everywhere and is commonly referred to in the literature as the confluent hypergeometric function or Kummer's function.

Many mathematicians extensively studied the hypergeometric functions ${}_2F_1$ and ${}_3F_2$ due to their wide variety of uses in various fields such as mathematical physics, number theory, and combinatorics [18, 20, 25].

By specifying particular values for the parameters and the arguments in ${}_2F_1$ and ${}_3F_2$, summation theorems yield explicit expressions for these hypergeometric functions, often in terms of Gamma functions. The hypergeometric functions ${}_2F_1$ and ${}_3F_2$ are linked to several classical summation theorems, including Gauss's theorem for ${}_2F_1$ at $z = 1$, Gauss's second and Bailey's theorems for ${}_2F_1$ at $z = \frac{1}{2}$, and Kummer's theorem for ${}_2F_1$ at $z = -1$. Additionally, Watson's, Dixon's, Whipple's, and Pfaff-Saalschütz's theorems apply to Clausen's series ${}_3F_2(1)$ with specific parameters, while Second Whipple's theorem addresses ${}_4F_3$ at $z = -1$. For more details on these summation theorems, including detailed formulations and proofs, refer to Koepf et al. [11, p.108] and Andrews et al. [2]. It is important to point out here that, whenever hypergeometric functions, whether they are the basic ${}_2F_1$ or the more generalized ${}_pF_q$, reduce to well-known functions like gamma functions, it simplifies mathematical expressions, making them more manageable for additional analysis and computation.

Remark 1. *For interesting generalizations and extensions of the long-established summation theorems described above, we refer to relevant papers by* [13, 14, 15], [8], *and* [24].

Another form of hypergeometric series, known as finite sums of hypergeometric series, is defined by the following symbol (see [17]):

$$ {}_p\overset{(k)}{F}_q \left[\begin{array}{ccc} \rho_1, & \cdots, & \rho_p \\ \beta_1, & \cdots, & \beta_q \end{array} \mid z \right] = \sum_{n=0}^{k} \frac{\prod_{i=1}^{p} (\rho_i)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!}, \tag{1.3} $$

It is evident that the generalized hypergeometric series (1.2) encompasses all integer values of n from 0 to infinity, characterized by its hypergeometric nature. In contrast, when the upper limit of the summation is a finite natural number k, the series becomes a finite sum of the first $k+1$ terms on the right-hand side of (1.2). As an illustration, we consider particular cases when $k = -1, 0$, and 1, which yield different results and provide insights into specific instances of finite hypergeometric series. For $k = -1$, the hypergeometric series is an empty sum, resulting in ${}_p\overset{(-1)}{F}_q(z) = 0$, and for $k = 0$, it consists of only one term, equaling ${}_p\overset{(0)}{F}_q(z) = 1$. For $k = 1$, the series includes two terms, giving ${}_p\overset{(1)}{F}_q(z) = 1 + \frac{\rho_1 \cdots \rho_p}{\beta_1 \cdots \beta_q} z$.

By using the following relation mentioned in [22],

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}\rho_1,\cdots,\rho_{p-1},1\\ \beta_1,\cdots,\beta_{q-1},k\end{matrix}\mid z\right]\\
&=\frac{\prod_{j=1}^{q-1}\Gamma(\beta_j)}{\prod_{i=1}^{p-1}\Gamma(\rho_i)}\cdot\frac{\prod_{i=1}^{p-1}\Gamma(\rho_i-k+1)}{\prod_{j=1}^{q-1}\Gamma(\beta_j-k+1)}\frac{(k-1)!}{z^{k-1}}\\
&\times\left\{{}_{p-1}F_{q-1}\left[\begin{matrix}\rho_1-k+1,\cdots,\rho_{p-1}-k+1\\ \beta_1-k+1,\cdots,\beta_{q-1}-k+1\end{matrix}\mid z\right]\right.\\
&\left.-\,{}_{p-1}^{(k-2)}F_{q-1}\left[\begin{matrix}\rho_1-k+1,\cdots,\rho_{p-1}-k+1\\ \beta_1-k+1,\cdots,\beta_{q-1}-k+1\end{matrix}\mid z\right]\right\},
\end{aligned}
\tag{1.4}
$$

Significant extensions of the well-known summation formulae, ranging from Gauss to Whipple, have been provided recently by Masjed-Jamei and Koepf [17]. For details on the original expressions of these classical summation formulae, readers are directed to the comprehensive references [11, 2, 4], where the foundational results are discussed in detail. The extended results, presented in terms of Gamma functions and finite sums of hypergeometric series, are provided in equations (1.5)-(1.11) below:

(i) The extended version of the Gauss summation theorem, as provided by Masjed-Jamei and Koepf [17], is given by:

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\rho,\ \beta,\ 1\\ \gamma,\ \ k\end{matrix}\mid 1\right]&=\frac{\Gamma(k)\,\Gamma(\gamma)\,\Gamma(1+\rho-k)\,\Gamma(1+\beta-k)}{\Gamma(\beta)\,\Gamma(\rho)\,\Gamma(1+\gamma-k)}\\
&\times\left\{\frac{\Gamma(\gamma-k+1)\,\Gamma(\gamma-\rho-\beta+k-1)}{\Gamma(\gamma-\beta)\ \Gamma(\gamma-\rho)}\right.\\
&\left.-\,{}_2^{(k-2)}F_1\left[\begin{matrix}\rho-k+1,\ \beta-k+1\\ \gamma-k+1\end{matrix}\mid 1\right]\right\}\\
&=\Theta_1\ (\text{let})
\end{aligned}
\tag{1.5}
$$

(ii) The extended version of Kummer's summation formula, as discussed in Masjed-Jamei and Koepf [17], is given by:

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\rho,\ \ \beta,\ \ 1\\ \rho+k-\beta,\ \ k\end{matrix}\mid -1\right]\\
&=(-1)^{k-1}\frac{\Gamma(k)\,\Gamma(\rho+k-\beta)\,\Gamma(\rho-k+1)\,\Gamma(\beta-k+1)}{\Gamma(\beta)\,\Gamma(\rho)\,\Gamma(\rho+1-\beta)}\\
&\times\left\{\frac{\Gamma(\rho+1-\beta)\ \Gamma\left(\frac{\rho-k+1}{2}+1\right)}{\Gamma(\rho-k+2)\,\Gamma\left(\frac{\rho-k+1}{2}+k-\beta\right)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\left. - {}_2F_1^{(k-2)} \left[\begin{matrix} \rho - k + 1, & \beta - k + 1 \\ \rho - \beta + 1 & \end{matrix} \;\middle|\; -1 \right] \right\} \\
&= \Theta_2 \text{ (let)}
\end{aligned} \tag{1.6}
$$

(iii) The extended version of the second kind of Gauss formula, as expressed in Masjed-Jamei and Koepf [17], is given by:

$$
\begin{aligned}
&{}_3F_2 \left[\begin{matrix} \rho, & \beta, & 1 \\ \frac{\rho+\beta+1}{2}, & k & \end{matrix} \;\middle|\; \frac{1}{2} \right] \\
&= 2^{k-1} \frac{\Gamma(k)\,\Gamma\left(\frac{\rho+\beta+1}{2}\right)\Gamma(\beta-k+1)\,\Gamma(\rho-k+1)}{\Gamma(\rho)\,\Gamma(\beta)\,\Gamma\left(-k+\frac{\rho+\beta+1}{2}+1\right)} \\
&\times \left\{ \frac{\sqrt{\pi}\,\Gamma\left(-k+\frac{\rho+\beta+1}{2}+1\right)}{\Gamma\left(\frac{\rho-k}{2}+1\right)\Gamma\left(\frac{\beta-k}{2}+1\right)} \right. \\
&\left. - {}_2F_1^{(k-2)} \left[\begin{matrix} \rho - k + 1, & \beta - k + 1 \\ -k + \frac{\rho+\beta+1}{2} + 1 & \end{matrix} \;\middle|\; \frac{1}{2} \right] \right\} \qquad = \Theta_3 \text{ (let)}
\end{aligned} \tag{1.7}
$$

(iv) The extension of the Bailey formula, as expressed in Masjed-Jamei and Koepf [17], is given by :

$$
\begin{aligned}
{}_3F_2 \left[\begin{matrix} \rho, & 2k-\rho-1, & 1 \\ & \beta, & k \end{matrix} \;\middle|\; \frac{1}{2} \right] &= 2^{k-1} \frac{\Gamma(\beta)\,\Gamma(k)\,\Gamma(\rho-k+1)\,\Gamma(k-\rho)}{\Gamma(\rho)\,\Gamma(2k-\rho-1)\,\Gamma(\beta-k+1)} \\
&\times \left\{ \frac{\Gamma\left(\frac{\beta-k+1}{2}\right)\Gamma\left(\frac{\beta-k+2}{2}\right)}{\Gamma\left(\frac{\rho+\beta}{2}-k+1\right)\Gamma\left(\frac{\beta-\rho+1}{2}\right)} \right. \\
&\left. - {}_2F_1^{(k-2)} \left[\begin{matrix} \rho - k + 1, & k - \rho \\ \beta - k + 1 & \end{matrix} \;\middle|\; \frac{1}{2} \right] \right\} \\
&= \Theta_4 \text{ (let)}
\end{aligned} \tag{1.8}
$$

(v) The extension of Dixon's theorem, as provided by Masjed-Jamei and Koepf [17], is given by:

$$
\begin{aligned}
&{}_4F_3 \left[\begin{matrix} \rho, & \beta, & \gamma, & 1 \\ \rho-\beta+k, & \rho-\gamma+k, & k & \end{matrix} \;\middle|\; 1 \right] \\
&= \frac{\Gamma(k)\,\Gamma(\rho-\beta+k)\,\Gamma(\rho-\gamma+k)\,\Gamma(\rho+1-k)\,\Gamma(\beta+1-k)\,\Gamma(\gamma+1-k)}{\Gamma(\gamma)\,\Gamma(\beta)\,\Gamma(\rho)\,\Gamma(\rho-\beta+1)\,\Gamma(\rho-\gamma+1)} \\
&\times \left\{ \frac{\Gamma\left(\frac{\rho+3-k}{2}\right)\Gamma(\rho-\beta+1)\,\Gamma(\rho-\gamma+1)\,\Gamma\left(\frac{\rho+3k-1}{2}-\beta-\gamma\right)}{\Gamma(\rho+2-k)\,\Gamma\left(\frac{\rho+k+1}{2}-\beta\right)\Gamma\left(\frac{\rho+k+1}{2}-\gamma\right)\Gamma(\rho-\beta-\gamma+k)} \right.
\end{aligned}
$$

$$- {}_3F_2^{(k-2)} \left[\begin{matrix} \rho-k+1, & \beta-k+1, & \gamma-k+1 \\ 1+\rho-\beta, & 1+\rho-\gamma & \end{matrix} \middle| 1 \right] \Bigg\} \tag{1.9}$$
$$= \Theta_5 \text{ (let)}$$

(vi) The extension of Watson's theorem, as represented by Masjed-Jamei and Koepf [17], is given by:

$$\begin{aligned} &{}_4F_3 \left[\begin{matrix} \rho, \ \beta, \ \gamma, \ 1 \\ \frac{\rho+\beta+1}{2}, \ 2\gamma+1-k, \ k \end{matrix} \middle| 1 \right] \\ &= \frac{\Gamma(k)\Gamma\left(\frac{\rho+\beta+1}{2}\right)\Gamma(2\gamma-k+1)\Gamma(\rho-k+1)\Gamma(\beta-k+1)\Gamma(\gamma-k+1)}{\Gamma(\rho)\Gamma(\beta)\Gamma(\gamma)\Gamma\left(\frac{\rho+\beta+3}{2}-k\right)\Gamma(2\gamma-2k+2)} \\ &\times \Bigg\{ \frac{\sqrt{\pi}\ \ \Gamma\left(\gamma-k+\frac{3}{2}\right)\Gamma\left(\frac{\rho+\beta+3}{2}-k\right)\Gamma\left(\gamma-\frac{\rho+\beta-1}{2}\right)}{\Gamma\left(\frac{\rho-k}{2}+1\right)\Gamma\left(\frac{\beta-k}{2}+1\right)\Gamma\left(\gamma-\frac{\rho+k}{2}+1\right)\Gamma\left(\gamma-\frac{\beta+k}{2}+1\right)} \\ &- {}_3F_2^{(k-2)} \left[\begin{matrix} \rho-k+1, \ \beta-k+1, \ \gamma-k+1 \\ -k+\frac{\rho+\beta+1}{2}+1, \ 2\gamma-2k+2 \end{matrix} \middle| 1 \right] \Bigg\} \\ &= \Theta_6 \text{ (let)} \end{aligned} \tag{1.10}$$

(vii) The extension of Whipple's theorem, as represented by Masjed-Jamei and Koepf [17], is given by:

$$\begin{aligned} &{}_4F_3 \left[\begin{matrix} \rho, \ 2k-1-\rho, \ \beta, \ 1 \\ \gamma, \ 2\beta-\gamma+1, \ k \end{matrix} \middle| 1 \right] \\ &= \frac{\Gamma(k)\Gamma(\gamma)\Gamma(2\beta-\gamma+1)\Gamma(k-\rho)\Gamma(\beta-k+1)\Gamma(\rho-k+1)}{\Gamma(\beta)\Gamma(\rho)\Gamma(2k-\rho-1)\Gamma(\gamma-k+1)\Gamma(2\beta-\gamma-k+2)} \\ &\times \Bigg\{ \frac{\pi\ 2^{2k-2\beta-1}\ \Gamma(\gamma-k+1)}{\Gamma\left(\beta+\frac{1+\rho-\gamma}{2}-k+1\right)\Gamma\left(\frac{\rho+\gamma}{2}-k+1\right)\Gamma\left(\frac{1-\rho+\gamma}{2}\right)} \\ &\times \frac{\Gamma(2\beta-\gamma-k+2)}{\Gamma\left(\beta-\frac{\rho+\gamma}{2}+1\right)} - {}_3F_2^{(k-2)} \left[\begin{matrix} \rho-k+1, \ \beta-k+1, \ k-\rho \\ \gamma-k+1, \ 2\beta-\gamma-k+2 \end{matrix} \middle| 1 \right] \Bigg\} \\ &= \Theta_7 \text{ (let)} \end{aligned} \tag{1.11}$$

In 1961, MacRobert [16] evaluated the following integral and obtained the answer in terms of Gamma functions.

$$\int_0^1 t^{a-1}(1-t)^{b-1}[rt+s(1-t)]^{-a-b}dt = \frac{1}{r^a s^b}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{1.12}$$

provided $Re(a) > 0$, $Re(b) > 0$ and r and s are non-zero constants and the expression $rt + s(1-t)$ is non-zero for all $t \in [0,1]$.
In recent years, several authors have investigated MacRobert-style integral formulas involving various generalizations of classical special functions. For instance, in 2019, Khan et al. [7] studied such integrals involving Bessel-Struve kernel function, which are expressed in terms of Wright generalized hypergeometric functions and then transformed into generalized hypergeometric functions. In 2018, Kim and co-authors [9] established several MacRobert-style generalized integrals involving the hypergeometric function ${}_3F_2$. Later, in 2020, Kim [10] independently derived some integrals involving the function ${}_4F_3$. These results were derived using an extension of Watson's summation theorem developed by Lavoie et al. [13]. Furthermore, in 2021, Jatav and Shukla [6] discussed MacRobert-style integrals associated with a general class of polynomials defined by Prabhakar and Suman [21], expressed in terms of the ${}_pR_q(\tau, \mu; z)$ function introduced by Desai and Shukla [5]. In 2024, Mishra et al. [19] derived some MacRobert-style integral formulas combining the k-Struve and Mittag-Leffler functions. These works demonstrate the broad applicability of the MacRobert-style integral approach across various special functions and within different mathematical contexts.

The novelty of our work lies in the derivation of a new class of seven MacRobert-style integrals incorporating generalized hypergeometric functions, utilizing the suitable extended versions of summation formulas for ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_6F_5$ given by (1.5) through (1.11) which are provided by Masjed-Jamei and Koepf [17]. These results generalize and unify by encompassing several known integrals as special cases. Moreover, the specific approach used allows for the inclusion of additional parameters and function classes, thus extending the applicability of the MacRobert integral framework to wider analytical contexts.

2. MacRobert - style integrals

The following theorems introduce seven new MacRobert-style integrals developed in this study by incorporating generalized hypergeometric functions.

Theorem 2.1. *Let $k \in \mathbb{N}, Re(\nu) > 0, Re(e-\nu) > 0$, and r and s be non-zero constants. If $rt + s(1-t) \neq 0$ for all $t \in [0,1]$, the following result holds.*

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^e}\, {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ e,\ 1\\ \gamma,\ \nu,\ k\end{matrix} \mid \frac{rt}{rt+s(1-t)}\right]dt \\ &= \frac{1}{r^\nu s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\Theta_1\end{aligned} \tag{2.1}$$

where Θ_1 is the same value as in (1.5).

Proof. Let I represent the left-hand side of (2.1). We will express ${}_4F_3$ in series form, switch the sequence of integration and summation, and then evaluate the MacRobert integral utilizing the result (1.12), we have

$$I = \sum_{n=0}^{\infty} \frac{(\rho)_n(\beta)_n(e)_n(1)_n r^n}{(\gamma)_n(\nu)_n(k)_n n!} \frac{1}{r^{\nu+n}s^{e-\nu}} \frac{\Gamma(\nu+n)\Gamma(e-\nu)}{\Gamma(e+n)}$$

Upon applying the ascending factorial (1.1) and simplifying, the result is obtained as

$$I = \sum_{n=0}^{\infty} \frac{(\rho)_n(\beta)_n(e)_n(1)_n r^n}{(\gamma)_n(\nu)_n(k)_n n!} \frac{1}{r^{\nu}r^n s^{e-\nu}} \frac{\Gamma(\nu)(\nu)_n\Gamma(e-\nu)}{\Gamma(e)(e)_n}$$

After summing the series, we arrive at

$$I = \frac{1}{r^{\nu}s^{e-\nu}} \frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)} \, {}_3F_2\left[\begin{matrix}\rho, \ \beta, \ 1 \\ \gamma, \ k\end{matrix} \mid 1\right]$$

Now, it is evident that the ${}_3F_2$ can be computed using the result (1.5), facilitating our arrival at the right side of (2.1).

Consequently, we have established the result in Theorem 2.1.

Corollary 2.1. *If we substitute $k=2$, and 3 (excluding the trivial case of $k=1$) into the preceding theorem, we respectively obtain the integrals presented below.*

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^e} {}_4F_3\left[\begin{matrix}\rho, \ \beta, \ e, \ 1 \\ \gamma, \ \nu, \ 2\end{matrix} \mid \frac{rt}{rt+s(1-t)}\right]dt \\ &= \frac{1}{r^{\nu}s^{e-\nu}} \frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)} \frac{(\gamma-1)}{(\beta-1)(\rho-1)} \left[\frac{\Gamma(\gamma-1)\Gamma(\gamma-\rho-\beta+1)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\rho)} - 1\right]\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^e} {}_4F_3\left[\begin{matrix}\rho, \ \beta, \ e, \ 1 \\ \gamma, \ \nu, \ 3\end{matrix} \mid \frac{rt}{rt+s(1-t)}\right]dt \\ &= \frac{1}{r^{\nu}s^{e-\nu}} \frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)} \frac{2(\gamma-2)_2}{(\rho-2)_2(\beta-2)_2} \\ &\times \left[\frac{\Gamma(\gamma-2)\Gamma(\gamma-\rho-\beta+2)}{\Gamma(\gamma-\rho)\Gamma(\gamma-\beta)} - \frac{\rho\beta+\gamma-2\rho-2\beta+2}{\gamma-2}\right]\end{aligned} \tag{2.3}$$

Similarly, the subsequent theorems and their associated corollaries can be derived by applying the results (1.6) to (1.11). Therefore, they are provided here without derivation.

Theorem 2.2. *Let $k \in \mathbb{N}, Re(e) > 0, Re(\nu - 2e + k) > 0,$ and r and s be non-zero constants. If $rt + s(1-t) \neq 0$ for all $t \in [0,1]$, the following result holds.*

$$\begin{aligned} &\int_0^1 \frac{t^{e-1}(1-t)^{\nu-2e+k-1}}{[rt+s(1-t)]^{\nu-e+k}} \; {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ \nu-e+k,\ 1 \\ \rho-\beta+k,\ e,\ k\end{matrix} \mid \frac{-rt}{rt+s(1-t)}\right]dt \\ &\quad = \frac{1}{r^e s^{\nu-2e+k}} \frac{\Gamma(e)\Gamma(\nu-2e+k)}{\Gamma(\nu-e+k)}\Theta_2 \end{aligned} \tag{2.4}$$

where Θ_2 is the same value as in (1.6).

Corollary 2.2. *If we substitute $k = 1, 2,$ and 3 into the preceding theorem, we respectively obtain the integrals presented below.*

$$\begin{aligned} &\int_0^1 \frac{t^{e-1}(1-t)^{\nu-2e}}{[rt+s(1-t)]^{\nu-e+1}} \; {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ \nu-e+1,\ 1 \\ \rho-\beta+1,\ e,\ 1\end{matrix} \mid \frac{-rt}{rt+s(1-t)}\right]dt \\ &\quad = \frac{1}{r^e s^{\nu-2e+1}} \frac{\Gamma(e)\Gamma(\nu-2e+1)}{\Gamma(\nu-e+1)} \frac{\Gamma(\rho-\beta+1)\Gamma(1+\frac{\rho}{2})}{\Gamma(\rho+1)\Gamma(1+\frac{\rho-\beta}{2})} \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\int_0^1 \frac{t^{e-1}(1-t)^{\nu-2e+1}}{[rt+s(1-t)]^{\nu-e+2}} \; {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ \nu-e+2,\ 1 \\ \rho-\beta+2,\ e,\ 2\end{matrix} \mid \frac{-rt}{rt+s(1-t)}\right]dt \\ &\quad = \frac{1}{r^e s^{\nu-2e+2}} \frac{\Gamma(e)\Gamma(\nu-2e+2)}{\Gamma(\nu-e+2)} \frac{(\rho-\beta+1)}{(\beta-1)(\rho-1)} \\ &\quad \times \left[1 - \frac{\Gamma(\rho-\beta+1)\Gamma(\frac{\rho+1}{2})}{\Gamma(\rho)\Gamma(\frac{\rho}{2}+\frac{3}{2}-\beta)}\right] \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\int_0^1 \frac{t^{e-1}(1-t)^{\nu-2e+2}}{[rt+s(1-t)]^{\nu-e+3}} \; {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ \nu-e+3,\ 1 \\ \rho-\beta+3,\ e,\ 3\end{matrix} \mid \frac{-rt}{rt+s(1-t)}\right]dt \\ &\quad = \frac{1}{r^e s^{\nu-2e+3}} \frac{\Gamma(e)\Gamma(\nu-2e+3)}{\Gamma(\nu-e+3)} \frac{2(\rho-\beta+1)_2}{(\beta-2)_2(\rho-2)_2} \\ &\quad \times \left[\frac{\Gamma(\rho-\beta+1)\Gamma(\frac{\rho}{2})}{\Gamma(\frac{\rho}{2}-\beta+2)\Gamma(\rho-1)} - \frac{3\rho+\beta-\rho\beta-3}{\rho-\beta+1}\right] \end{aligned} \tag{2.7}$$

Theorem 2.3. *Let $k \in \mathbb{N}, Re(e) > 0, Re(\nu - e + 1) > 0,$ and r and s be non-zero constants. If $rt + s(1-t) \neq 0$ for all $t \in [0,1]$, the following result holds.*

$$\begin{aligned} &\int_0^1 \frac{t^{e-1}(1-t)^{\frac{\nu-e-1}{2}}}{[rt+s(1-t)]^{\frac{(\nu+e+1)}{2}}} \; {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ \frac{\nu+e+1}{2},\ 1 \\ e,\ \frac{\rho+\beta+1}{2},\ k\end{matrix} \mid \frac{rt}{2(rt+s(1-t))}\right]dt \\ &\quad = \frac{1}{r^e s^{\frac{\nu-e+1}{2}}} \frac{\Gamma(e)\Gamma(\frac{\nu-e+1}{2})}{\Gamma(\frac{\nu+e+1}{2})}\Theta_3 \end{aligned} \tag{2.8}$$

where Θ_3 is the same value as in (1.7).

Corollary 2.3. *If we substitute $k = 1, 2,$ and 3 into the preceding theorem, we respectively obtain the integrals presented below.*

$$\begin{aligned}&\int_0^1 \frac{t^{e-1}(1-t)^{\frac{\nu-e-1}{2}}}{[rt+s(1-t)]^{\frac{1}{2}(\nu+e+1)}}\ {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ \frac{\nu+e+1}{2},\ 1\\ e,\ \frac{\rho+\beta+1}{2},\ 1\end{matrix} \mid \frac{rt}{2(rt+s(1-t))}\right]dt\\ &= \frac{\sqrt{\pi}\Gamma(e)}{r^e s^{\frac{\nu-e+1}{2}}}\frac{\Gamma(\frac{\nu-e+1}{2})}{\Gamma(\frac{\nu+e+1}{2})}\frac{\Gamma(\frac{\rho+1+\beta}{2})}{\Gamma(\frac{\beta+1}{2})\Gamma(\frac{\rho+1}{2})}\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\int_0^1 \frac{t^{e-1}(1-t)^{\frac{\nu-e-1}{2}}}{[rt+s(1-t)]^{\frac{1}{2}(\nu+e+1)}}\ {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ \frac{\nu+e+1}{2},\ 1\\ e,\ \frac{\rho+\beta+1}{2},\ 2\end{matrix} \mid \frac{rt}{2(rt+s(1-t))}\right]dt\\ &= \frac{1}{r^e s^{\frac{\nu-e+1}{2}}}\frac{\Gamma(e)\Gamma(\frac{\nu-e+1}{2})}{\Gamma(\frac{\nu+e+1}{2})}\frac{(\rho-1+\beta)}{(\beta-1)(\rho-1)}\\ &\times\left[\frac{\sqrt{\pi}\ \Gamma(\frac{\rho-1+\beta)}{2}}{\Gamma(\frac{\beta}{2}\ \Gamma(\frac{\rho}{2})}-1\right]\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\int_0^1 \frac{t^{e-1}(1-t)^{\frac{\nu-e-1}{2}}}{[rt+s(1-t)]^{\frac{1}{2}(\nu+e+1)}}\ {}_4F_3\left[\begin{matrix}\rho,\ \beta,\ \frac{\nu+e+1}{2},\ 1\\ e,\ \frac{\rho+\beta+1}{2},\ 3\end{matrix} \mid \frac{rt}{2(rt+s(1-t))}\right]dt\\ &= \frac{1}{r^e s^{\frac{\nu-e+1}{2}}}\frac{\Gamma(e)\Gamma(\frac{\nu-e+1}{2})}{\Gamma(\frac{\nu+e+1}{2})}\frac{2(\rho-1+\beta)(\rho-3+\beta)}{(\beta-2)_2\ (\rho-2)_2}\\ &\times\left[\frac{\sqrt{\pi}\ \Gamma(\frac{\beta+\rho}{2}-\frac{3}{2})}{\Gamma(\frac{\beta-1}{2})\Gamma(\frac{\rho-1}{2})}-\frac{(\rho\beta-\beta-\rho+1)}{(\rho-3+\beta)}\right]\end{aligned} \tag{2.11}$$

Theorem 2.4. *Let $k \in \mathbb{N}$, $Re(\nu) > 0, Re(e-\nu) > 0$, and r and s be non-zero constants. If $rt + s(1-t) \neq 0$ for all $t \in [0, 1]$, the following result holds.*

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^e}\ {}_4F_3\left[\begin{matrix}\rho,\ 2k-\rho-1,\ e,\ 1\\ \beta,\ \nu,\ k\end{matrix} \mid \frac{rt}{2(rt+s(1-t))}\right]dt\\ &= \frac{1}{r^\nu s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\Theta_4\end{aligned} \tag{2.12}$$

where Θ_4 is the same value as in (1.8).

Corollary 2.4. *If we substitute $k = 1, 2,$ and 3 into the preceding theorem, we* *respectively obtain the integrals presented below.*

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^e}\ {}_4F_3\left[\begin{matrix}\rho,\ 1-\rho,\ e,\ 1\\ \beta,\ \nu,\ 1\end{matrix} \mid \frac{rt}{2(rt+s(1-t))}\right]dt\\ &= \frac{1}{r^\nu s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta+1}{2})}{\Gamma(\frac{\rho+\beta}{2})\Gamma(\frac{\beta-\rho+1}{2})}\end{aligned} \tag{2.13}$$

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^e}\ {}_4F_3\left[\begin{matrix}\rho,\ 3-\rho,\ e,\ 1\\ \beta,\ \nu,\ 2\end{matrix} \mid \frac{rt}{2(rt+s(1-t))}\right]dt\\ &= \frac{1}{r^\nu s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\frac{2(1-\beta)}{(\rho-2)_2}\\ &\times\left[\frac{\Gamma(\frac{\beta-1}{2})\ \Gamma(\frac{\beta}{2})}{\Gamma(\frac{1-\rho+\beta}{2})\Gamma(\frac{\rho+\beta}{2}-1)}-1\right]\end{aligned} \tag{2.14}$$

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^e}\ {}_4F_3\left[\begin{matrix}\rho,\ 5-\rho,\ e,\ 1\\ \beta,\ \nu,\ 3\end{matrix} \mid \frac{rt}{2(rt+s(1-t))}\right]dt\\ &= \frac{1}{r^\nu s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\frac{8(\beta-2)_2}{(\rho-4)_4}\\ &\times\left[\frac{\Gamma(\frac{\beta-1}{2})\Gamma(\frac{\beta-2}{2})}{\Gamma(\frac{\beta-\rho+1}{2})\ \Gamma(\frac{\rho+\beta}{2}-2)}-\frac{5\rho-\rho^2+2\beta-10}{2(\beta-2)}\right]\end{aligned} \tag{2.15}$$

Theorem 2.5. *Let $k\in\mathbb{N}, Re(\nu)>0, Re(e-2\nu+k)>0$, and r and s be non-zero constants. If $rt+s(1-t)\neq 0$ for all $t\in[0,1]$, the following result holds.*

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-2\nu+k-1}}{[rt+s(1-t)]^{e-\nu+k}}\ {}_5F_4\left[\begin{matrix}\rho,\ \beta,\ \gamma,\ e-\nu+k,\ 1\\ \rho-\beta+k,\ \rho-\gamma+k,\ \nu,\ k\end{matrix} \mid \frac{rt}{rt+s(1-t)}\right]dt\\ &= \frac{1}{r^\nu s^{e-2\nu+k}}\frac{\Gamma(\nu)\Gamma(e-2\nu+k)}{\Gamma(e-\nu+k)}\Theta_5\end{aligned} \tag{2.16}$$

where Θ_5 is the same value as in (1.9).

Corollary 2.5. *If we substitute $k=1,2,$ and 3 into the preceding theorem, we respectively obtain the integrals presented below.*

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-2\nu}}{[rt+s(1-t)]^{e-\nu+1}}\ {}_5F_4\left[\begin{matrix}\rho,\ \beta,\ \gamma,\ e-\nu+1,\ 1\\ \rho-\beta+1,\ \rho-\gamma+1,\ \nu,\ 1\end{matrix} \mid \frac{rt}{rt+s(1-t)}\right]dt\\ &= \frac{1}{r^\nu s^{e-2\nu+1}}\frac{\Gamma(\nu)\Gamma(e-2\nu+1)}{\Gamma(e-\nu+1)}\\ &\times\frac{\Gamma(\rho-\gamma+1)\Gamma(\frac{\rho}{2}+1)\Gamma(\rho-\beta+1)\Gamma(\frac{\rho}{2}-\beta-\gamma+1)}{\Gamma(\rho+1)\Gamma(1+\frac{\rho}{2}-\beta)\Gamma(1+\frac{\rho}{2}-\gamma)\Gamma(\rho-\beta-\gamma+1)}\end{aligned} \tag{2.17}$$

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-2\nu+1}}{[rt+s(1-t)]^{e-\nu+2}}\,{}_5F_4\left[\begin{matrix}\rho,\ \beta,\ \gamma,\ e-\nu+2,\ 1\\ \rho-\beta+2,\ \rho-\gamma+2,\ \nu,\ 2\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt\\ &\quad=\frac{1}{r^{\nu}s^{e-2\nu+2}}\frac{\Gamma(\nu)\Gamma(e-2\nu+2)}{\Gamma(e-\nu+2)}\frac{(\rho-\beta+1)(\rho-\gamma+1)}{(\gamma-1)(\beta-1)(\rho-1)}\\ &\quad\left[\frac{\Gamma(\frac{\rho+1}{2})\Gamma(\rho-\gamma+1)\Gamma(\rho-\beta+1)\Gamma(\frac{\rho+5}{2}-\beta-\gamma)}{\Gamma(\rho)\Gamma(\frac{\rho+3}{2}-\beta)\Gamma(\frac{\rho+3}{2}-\gamma)\Gamma(\rho-\beta-\gamma+2)}-1\right]\end{aligned}\tag{2.18}$$

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{e-2\nu+2}}{[rt+s(1-t)]^{e-\nu+3}}\,{}_5F_4\left[\begin{matrix}\rho,\ \beta,\ \gamma,\ e-\nu+3,\ 1\\ \rho-\beta+3,\ \rho-\gamma+3,\ \nu,\ 3\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt\\ &\quad=\frac{1}{r^{\nu}s^{e-2\nu+3}}\frac{\Gamma(\nu)\Gamma(e-2\nu+3)}{\Gamma(e-\nu+3)}\frac{2(\rho-\beta+1)_2(\rho-\gamma+1)_2}{(\gamma-2)_2(\beta-2)_2(\rho-2)_2}\\ &\quad\times\left[\frac{\Gamma(\frac{\rho}{2})\Gamma(\rho-\gamma+1)\Gamma(\rho-\beta+1)\Gamma(\frac{\rho+8}{2}-\beta-\gamma)}{\Gamma(\rho-1)\Gamma(\frac{\rho}{2}-\gamma+2)\Gamma(\frac{\rho}{2}-\beta+2)\Gamma(\rho-\beta-\gamma+3)}\right.\\ &\quad\left.-\frac{(\gamma-2)(\beta-2)(\rho-2)}{(\rho-\gamma+1)(\rho-\beta+1)}-1\right]\end{aligned}\tag{2.19}$$

Theorem 2.6. *Let $k\in\mathbb{N}, Re(\nu)>0, Re(\nu-k+1)>0, Re(\nu-\rho-\beta+k)>1$, and r and s be non-zero constants. If $rt+s(1-t)\neq 0$ for all $t\in[0,1]$, the following result holds.*

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{\nu-k}}{[rt+s(1-t)]^{2\nu-k+1}}\,{}_5F_4\left[\begin{matrix}\rho,\ \beta,\ \gamma,\ 2\nu-k+1,\ 1\\ \frac{\rho+\beta+1}{2},\ 2\gamma-k+1,\ \nu,\ k\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt\\ &\quad=\frac{1}{r^{\nu}s^{\nu-k+1}}\frac{\Gamma(\nu)\Gamma(\nu-k+1)}{\Gamma(2\nu-k+1)}\Theta_6\end{aligned}\tag{2.20}$$

where Θ_6 is the same value as in (1.10).

Corollary 2.6. *If we substitute $k=1,2,$ and 3 into the preceding theorem, we respectively obtain the integrals presented below.*

$$\begin{aligned}&\int_0^1 \frac{t^{\nu-1}(1-t)^{\nu-1}}{[rt+s(1-t)]^{2\nu}}\,{}_5F_4\left[\begin{matrix}\rho,\ \beta,\ \gamma,\ 2\nu,\ 1\\ \frac{\rho+\beta+1}{2},\ 2\gamma,\ \nu,\ 1\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt\\ &\quad=\frac{1}{r^{\nu}s^{\nu}}\frac{\Gamma(\nu)\Gamma(\nu)}{\Gamma(2\nu)}\ \frac{\Gamma(\frac{\rho+\beta+1}{2})\sqrt{(\pi)}\Gamma(\gamma+\frac{1}{2})\Gamma(\gamma-\frac{\rho+\beta-1}{2})}{\Gamma(\frac{\rho+1}{2})\Gamma(\frac{\beta+1}{2})\Gamma(\gamma+\frac{1-\rho}{2}\Gamma(\gamma+\frac{1-\beta}{2})}\end{aligned}\tag{2.21}$$

$$\int_0^1 \frac{t^{\nu-1}(1-t)^{\nu-2}}{[rt+s(1-t)]^{2\nu-1}}\ {}_5F_4\left[\begin{matrix}\rho,\ \beta,\ \gamma,\ 2\nu-1,\ 1\\ \frac{\rho+\beta+1}{2},\ 2\gamma-1,\ \nu,\ 2\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt$$
$$= \frac{1}{r^{\nu}s^{\nu-1}}\frac{\Gamma(\nu)\Gamma(\nu-1)}{\Gamma(2\nu-1)}\frac{(\rho-1+\beta)}{(\beta-1)(\rho-1)} \tag{2.22}$$
$$\times\left[\frac{\sqrt{\pi}\Gamma(\gamma-\frac{1}{2})\Gamma(\frac{\rho-1+\beta}{2})\Gamma(\gamma-\frac{\rho-1+\beta}{2})}{\Gamma(\frac{\beta}{2})\Gamma(\gamma-\frac{\beta}{2})\Gamma(\frac{\rho}{2})\Gamma(\gamma-\frac{\rho}{2})}-1\right]$$

$$\int_0^1 \frac{t^{\nu-1}(1-t)^{\nu-3}}{[rt+s(1-t)]^{2\nu-2}}\ {}_5F_4\left[\begin{matrix}\rho,\ \beta,\ \gamma,\ 2\nu-2,\ 1\\ \frac{1+\rho+\beta}{2},\ 2\gamma-2,\ \nu,\ 3\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt$$
$$= \frac{1}{r^{\nu}s^{\nu-2}}\frac{\Gamma(\nu)\Gamma(\nu-2)}{\Gamma(2\nu-2)}\frac{(\rho-1+\beta)(\rho-3+\beta)(2\gamma-3)}{(\gamma-1)(\beta-2)_2\ (\rho-2)_2} \tag{2.23}$$
$$\times\left[\frac{\sqrt{\pi}\Gamma(\gamma-\frac{3}{2})\Gamma(\frac{\rho-3+\beta}{2})\Gamma(\gamma-\frac{\rho-1+\beta}{2})}{\Gamma(\gamma-\frac{\beta+1}{2})\Gamma(\gamma-\frac{\rho+1}{2})\Gamma(\frac{\beta-1}{2})\Gamma(\frac{\rho-1}{2})}-\frac{(\beta-2)(\rho-2)}{(\rho-3+\beta)}-1\right]$$

Theorem 2.7. *Let $k\in\mathbb{N}, Re(\nu)>0, Re(e-\nu)>0$, and r and s be non-zero constants. If $rt+s(1-t)\neq 0$ for all $t\in[0,1]$, the following result holds.*

$$\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^{e}}\ {}_5F_4\left[\begin{matrix}\rho,\ \beta,\ 2k-\rho-1,\ e,\ 1\\ 2\beta-\gamma+1,\ \gamma,\ \nu,\ k\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt$$
$$= \frac{1}{r^{\nu}s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\Theta_7 \tag{2.24}$$

where Θ_7 is the same value as in (1.11).

Corollary 2.7. *If we substitute $k=1,2,$ and 3 into the preceding theorem, we respectively obtain the integrals presented below.*

$$\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^{e}}\ {}_5F_4\left[\begin{matrix}\rho,\ \beta,\ 1-\rho,\ e,\ 1\\ 2\beta-\gamma+1,\ \gamma,\ \nu,\ 1\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt$$
$$= \frac{1}{r^{\nu}s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\ \frac{\Gamma(\gamma)\Gamma(2\beta-\gamma+1)\pi\ \ 2^{1-2\beta}}{\Gamma(\frac{\rho+\gamma}{2})\Gamma(\frac{\rho-\gamma+1}{2}+\beta)\Gamma(\beta-\frac{\rho+\gamma}{2}+1)\Gamma(\frac{1-\rho+\gamma}{2})} \tag{2.25}$$

$$\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^{e}}\ {}_5F_4\left[\begin{matrix}\rho,\ \beta,\ 3-\rho,\ e,\ 1\\ 2\beta-\gamma+1,\ \gamma,\ \nu,\ 2\end{matrix}\mid \frac{rt}{rt+s(1-t)}\right]dt$$
$$= \frac{1}{r^{\nu}s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\frac{(\gamma-1)(\gamma-2\beta)}{(\rho-2)_2(\beta-1)} \tag{2.26}$$
$$\left[\frac{\pi\ 2^{3-2\beta}\ \Gamma(2\beta-\gamma)\ \Gamma(\gamma-1)}{\Gamma(\frac{\rho+\gamma}{2}-1)\Gamma(\frac{\rho-\gamma+1}{2}-1+\beta)\Gamma(\frac{1-\rho+\gamma}{2})\Gamma(\beta-\frac{\rho+\gamma}{2}+1)}-1\right]$$

$$\int_0^1 \frac{t^{\nu-1}(1-t)^{e-\nu-1}}{[rt+s(1-t)]^e}\ {}_5F_4\left[\begin{matrix}\rho,\ \beta,\ 5-\rho,\ e,\ 1\\ 2\beta-\gamma+1,\ \gamma,\ \nu,\ 3\end{matrix}\ \Big|\ \frac{rt}{rt+s(1-t)}\right]dt$$
$$= \frac{1}{r^\nu s^{e-\nu}}\frac{\Gamma(\nu)\Gamma(e-\nu)}{\Gamma(e)}\frac{2(\gamma-2)_2\Gamma(2\beta-\gamma+1)}{\Gamma(2\beta-\gamma-1)(\rho-4)_2(\beta-2)_2}$$
$$\left[\frac{\pi\ 2^{5-2\beta}\Gamma(2\beta-\gamma-1)\Gamma(\gamma-2)}{\Gamma(\frac{\rho+\gamma}{2}-2)\Gamma(\beta+\frac{\rho-\gamma-3}{2})\Gamma(\frac{1-\rho+\gamma}{2})\Gamma(\beta+1-\frac{\rho+\gamma}{2})}\right. \tag{2.27}$$
$$\left. -1-\frac{(\beta-2)(\rho-2)(3-\rho)}{(\gamma-2)(2\beta-\gamma-1)}\right]$$

3. Special Cases

(i) If we take $\nu=\beta$, $e=\gamma$, $r=1+p$ and $s=1+q$, then the hypergeometric function ${}_4F_3$ used in (2.1) reduces to ${}_2F_1$ and we obtain a result previously established by Kulkarni et al. [12].

(ii) In the results (2.4) to (2.12), if we take $\nu=\rho$, $e=\beta$, $r=1+p$ and $s=1+q$, then the hypergeometric function ${}_4F_3$ reduces to ${}_2F_1$ and we obtain the corresponding results established by Kulkarni et al. [12].

Similarly, we can obtain other results.

4. Conclusion

In this paper, seven MacRobert-style integrals comprising generalized hypergeometric functions are expressed in terms of the Gamma function using Masjed-Jamei and Koepf's summation theorems. Also, a few well-established and completely new integrals have been included as particular instances associated with our key results.

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