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(m, n)-FUZZY DISTANCE MEASURES AND THEIR APPLICATIONS TO PATTERN RECOGNITION PROBLEMS

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Abstract: The (m,n)-fuzzy sets are an effective and efficient tool for depicting vagueness and uncertainty in information in decision making. The present paper created logarithmic and tangent inverse distance measures for (m,n)-FSs and explores some of their properties.Numerical examples are presented to show the validity and effectiveness of proposed distance measures.

Keywords and Phrases: (m, n)-fuzzy sets, distance measure of (m, n)-fuzzy sets, pattern recognition.

2020 Mathematics Subject Classification: 03E72, 68T10, 90B50.

1. Introduction

Traditional decision-making approaches based on classical set theory often struggle to manage incomplete, vague, or inconsistent information. To address such limitations, Zadeh [30] introduced fuzzy sets (FSs), which generalize classical sets to better accommodate imprecision and ambiguity. Since their introduction, FS theory has been widely employed across domains such as Science, Engineering, and Management. Researchers including Dubois and Prade [6], Rosenfeld [20], Adlassnig [1], Gerla and Volpe [8], Liu [14], Lee, Pedrycz, Sohn [12], and others have developed numerous distance measures for fuzzy objects and applied them in fields such as object recognition and medical diagnosis.

Building upon FSs, Atanassov [4] introduced intuitionistic fuzzy sets (IFSs), which assign to each element a membership degree (MD) and a non-membership degree (NMD), constrained such that their sum lies within the unit interval. This formulation spurred the development of various distance measures for IFSs. For example, Szmidt and Kacprzyk [23] proposed Euclidean and Hamming-based metrics, while Vlachos and Sergiadis [26] introduced entropy-based distances. Wang and Xin [27] further advanced this field with an axiomatic definition and a novel distance measure applied to pattern recognition.

Yager [28] later introduced Pythagorean fuzzy sets (PFSs) as an extension of IFSs, where the sum of the squares of MD and NMD must remain within $[0, 1]$. This was followed by the development of q-rung orthopair fuzzy sets (q-ROFSs) [29], in which the q^{th} power of MD and NMD lies within the same interval, allowing for greater flexibility in handling uncertainty. Fermatean fuzzy sets (FFSs), proposed by Senapati and Yager [21], represent a special case of q-ROFSs for $q = 3$. In recent years, scholars such as Aydın [5], Kirişci [11], Ashraf et al. [3], Peng et al. [16, 17], Zeng et al. [31], Hussian and Yang [9], and Ejegwa et al. [7] have introduced additional distance measures for PFSs, FFSs, and q-ROFSs, applying them to areas such as image processing, medical diagnostics, and decision-making.

One limitation common to these generalizations is their symmetric treatment of MD and NMD, which restricts flexibility in decision-making scenarios that require different weights or powers for MD and NMD. To address this, Al-Shami [2] proposed the concept of (m, n)-fuzzy sets ((m, n)-FSs), where the sum of the m^{th} power of MD and the n^{th} power of NMD must lie within $[0, 1]$. This model encompasses all n-ROFSs as a special case and proves more adaptable in multi-attribute decision-making (MADM) problems. Thakur et al. [25] introduced Hamming and Euclidean distance measures for (m, n)-FSs. Rajput, Shukla, and Thakur [19] developed cosine and cotangent similarity measures, which they applied to plant leaf disease classification. Additional metrics were proposed by Shivdas and John [24], who used t-norms in the context of lung disease diagnosis, and by Rahim et al. [18], who defined cosine-based distance functions.

This paper presents two novel distance measures for (m, n)-FSs, referred to as the (m, n)-fuzzy logarithmic distance and the (m, n)-fuzzy tangent inverse distance. Their properties are explored and validated through applications in construction material classification and plant leaf disease identification.

2. Preliminaries

Definition 2.1. [10, 2] *An structure* $\mathscr{G} = \{< p, \varrho_{\mathscr{G}}(p), \sigma_{\mathscr{G}}(p) >: p \in \mathbb{P}\}$ *where,* $\varrho_{\mathscr{G}} : \mathbb{P} \to [0, 1]$ *and* $\sigma_{\mathscr{G}} : \mathbb{P} \to [0, 1]$ *denotes the degree of membership and the degree of nonmembership of each* $p \in \mathbb{P}$ *to* $\mathscr{G}$ *is called a (m,n)- fuzzy set ((m,n)-FS) where* $m, n \in \mathbb{N}$ *in* $\mathbb{P}$ *if* $0 \leq \varrho^m_{\mathscr{G}}(p) + \sigma^n_{\mathscr{G}}(p) \leq 1, \ \forall \ p \in \mathbb{P}$.

For simplicity an (m,n)-FS $\mathscr{G} = \{< p, \varrho_{\mathscr{G}}(p), \sigma_{\mathscr{G}}(p) >: p \in \mathbb{P}\}$ will be denoted by $(\varrho_{\mathscr{G}}, \sigma_{\mathscr{G}})$ and $\mathcal{F}^n_m(\mathbb{P})$ refers the family of all (m,n)-FSs over $\mathbb{P}$.

Remark 2.2. [2] *Definition 2.1 can be reduced to the definition of :*

(i) q-ROFSs if m=n=q.

(ii) FFS if m=n=3.

(iii) PFS if m=n=2.

(iv) IFS if m=n=1.

(v) FS if m=1,n=0 .

Definition 2.3. [10] *Let* $\mathscr{G}, \mathscr{G}_1, \mathscr{G}_2 \in \mathcal{F}^n_m(\mathbb{P})$*. Then the subset, equality, union and intersection over* $\mathcal{F}^n_m(\mathbb{P})$ *are defined as follow:*

(a) $\mathscr{G}_1 \Subset \mathscr{G}_2 \Leftrightarrow \varrho_{\mathscr{G}_1} \leq \varrho_{\mathscr{G}_2}$ *and* $\sigma_{\mathscr{G}_1} \geq \sigma_{\mathscr{G}_2}$.

(b) $\mathscr{G}_1 = \mathscr{G}_2 \Leftrightarrow \varrho_{\mathscr{G}_1} = \varrho_{\mathscr{G}_2}$ *and* $\sigma_{\mathscr{G}_1} = \sigma_{\mathscr{G}_2}$.

(c) $\mathscr{G}_1 \Cup \mathscr{G}_2 = (max\{\varrho_{\mathscr{G}_1}, \ \varrho_{\mathscr{G}_2}\}, \ min\{\sigma_{\mathscr{G}_1}, \ \sigma_{\mathscr{G}_2}\})$.

(d) $\mathscr{G}_1 \Cap \mathscr{G}_2 = (min\{\varrho_{\mathscr{G}_1}, \ \varrho_{\mathscr{G}_2}\}, \ max\{\sigma_{\mathscr{G}_2}, \ \sigma_{\mathscr{G}_2}\})$.

(e) $\mathscr{G}^c = (\sigma_{\mathscr{G}}, \varrho_{\mathscr{G}})$.

Definition 2.4. [25] *Let* $\mathscr{G} \in \mathcal{F}^n_m(\mathbb{P})$ *and* $p \in \mathbb{P}$*. Then the degree of indeterminacy of* p *to* $\mathscr{G}$ *is defined as* $\pi_{\mathscr{G}}(p) = (1 - \varrho^m_{\mathscr{G}}(p) - \sigma^n_{\mathscr{G}}(p))^{\frac{2}{m+n}}$.

3. Novel distance measures for (m,n)-FSs

Recently, Thakur and his associates [25] proposed the axiomatic definition of distance measure for ((m,n)-FSs as follows:

Definition 3.1. [25] *Let* $\mathbb{P} = \{p_1, p_2, ...p_r.\}$ *be an universe of discourse and* $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3 \in \mathcal{F}_m^n(\mathbb{P})$, *the distance function* $d : \mathcal{F}_m^n(\mathbb{P}) \times \mathcal{F}_m^n(\mathbb{P}) \rightarrow [0, 1]$ *is defined as:*

(i) $0 \leq d(\mathscr{G}_1, \mathscr{G}_2) \leq 1$ *(boundedenes).*

(ii) $d(\mathscr{G}_1, \mathscr{G}_2) = 0 \Leftrightarrow \mathscr{G}_1 = \mathscr{G}_2$ *(separability).*

(iii) $d(\mathscr{G}_1, \mathscr{G}_2) = d(\mathscr{G}_2, \mathscr{G}_1)$ *(symmetric).*

(iv) $d(\mathscr{G}_1, \mathscr{G}_2) \leq d(\mathscr{G}_2, \mathscr{G}_3)$ *and* $d(\mathscr{G}_2, \mathscr{G}_3) \leq d(\mathscr{G}_1, \mathscr{G}_3)$ *if* $\mathscr{G}_1 \Subset \mathscr{G}_2 \Subset \mathscr{G}_3$ *(inequality).*

The (m,n)-fuzzy normalize Hamming distance and the (m,n)-fuzzy normalize Euclidean distance for(m,n)-FSs are proposed by Thakur et.al. [25].

Definition 3.2. [25] *Let* $\mathbb{P} = \{p_1, p_2, ...p_r.\}$ *and* $\mathscr{G}_1, \mathscr{G}_2 \in \mathcal{F}_m^n(\mathbb{P})$. *The (m,n)-fuzzy normalize Hamming distance* $d^{nH}_{F^n_m}$ *and the (m,n)-fuzzy normalized Euclidean distance* $d^{nE}_{F^n_m}$ *between* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *are respectively defined as follows:*

$$d^{nH}_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2) = \frac{1}{2r} \sum_{1}^{r} \begin{pmatrix} |\varrho_{\mathscr{G}_1}(p_i) - \varrho_{\mathscr{G}_2}(p_i)| \\ +|\sigma_{\mathscr{G}_1}(p_i) - \sigma_{\mathscr{G}_2}(p_i)| \\ +|\pi_{\mathscr{G}_1}(p_i) - \pi_{\mathscr{G}_2}(p_i)| \end{pmatrix}. \tag{1}$$

$$d^{nE}_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2) = \sqrt{\frac{1}{2r} \sum_{1}^{r} \begin{pmatrix} (\varrho_{\mathscr{G}_1}(p_i) - \varrho_{\mathscr{G}_2}(p_i))^2 \\ +(\sigma_{\mathscr{G}_1}(p_i) - \sigma_{\mathscr{G}_2}(p_i))^2 \\ +(\pi_{\mathscr{G}_1}(p_i) - \pi_{\mathscr{G}_2}(p_i))^2 \end{pmatrix}}. \tag{2}$$

In this section we proposed two new (m,n)-fuzzy distance measures for (m,n)-FSs and present their properties.

3.1. Logarithmic distance measure for (m,n)-FSs

Definition 3.3. *Let* $\mathbb{P} = \{p_1, p_2, ...p_r.\}$ *and* $\mathscr{G}_1, \mathscr{G}_2 \in \mathcal{F}_m^n(\mathbb{P})$. *The (m,n)- fuzzy logarithmic distance* $d^{L}_{F^n_m}$ *between* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *is defined as follows:*

$$d^{L}_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2) = \frac{1}{3rln2} \sum_{1}^{r} ln \begin{pmatrix} (1 + |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) \\ (1 + |\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)|) \\ (1 + |\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) \end{pmatrix}. \tag{3}$$

Theorem 3.4. *The (m,n)- fuzzy logarithmic distance measure $d^L_{F^n_m}$ between two(m,n)-FSs defined in definition 3.3 is a valid distance measure.*

Proof. We will show that the (m,n)- fuzzy logarithmic distance measure $d^L_{F^n_m}$ satisfies conditions (i)-(iv) of definition 3.1.

(i) Since $0 \le |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)| \le 1$, we have $1 \le (1 + |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) \le 2$. Similarly $1 \le (1+|\sigma^n_{\mathscr{G}_1}(p_i)-\sigma^n_{\mathscr{G}_2}(p_i)|) \le 2$, and $1 \le (1+|\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i)-\pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) \le 2$. Therefore,

$$\begin{aligned}
1 &\le \begin{pmatrix} (1 + |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) \\ (1 + |\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)|) \\ (1 + |\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) \end{pmatrix} \le 8 \\
\Rightarrow 0 &\le ln \begin{pmatrix} (1 + |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) \\ (1 + |\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)|) \\ (1 + |\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) \end{pmatrix} \le 3ln2 \\
\Rightarrow 0 &\le \frac{1}{3rln2} \sum_1^r ln \begin{pmatrix} (1 + |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) \\ (1 + |\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)|) \\ (1 + |\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) \end{pmatrix} \qquad \le 1.
\end{aligned}$$

Thus $0 \le d^L_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2) \le 1$.

(ii) If $\mathscr{G}_1 = \mathscr{G}_2$. Then $\varrho^m_{\mathscr{G}_1}(p_i) = \varrho^m_{\mathscr{G}_2}(p_i)$, $\sigma^n_{\mathscr{G}_1}(p_i) = \sigma^n_{\mathscr{G}_2}(p_i)$ and $\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) = \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)$. It follows that $|\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)| = 0$, $|\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)| = 0$ and $|\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)| = 0$. And so,$1 + |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)| = 1$,$1 + |\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)| = 1$ and $1 + |\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)| = 1$. Therefore

$$\begin{aligned}
d^L_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2) &= \frac{1}{3rln2} \sum_1^r ln \begin{pmatrix} (1 + |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) \\ (1 + |\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)|) \\ (1 + |\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) \end{pmatrix} \\
&= \frac{1}{3rln2} \sum_1^r ln1 \\
&= 0.
\end{aligned}$$

Conversely, if $d^L_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2) = 0$, then we have

$$\frac{1}{3rln2}\sum_{1}^{r} ln \begin{pmatrix} (1+|\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) \\ (1+|\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)|) \\ (1+|\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) \end{pmatrix} = 0$$
$$\Rightarrow 1+|\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) = 1, (1+|\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)|)$$
$$= 1, (1+|\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) = 1.$$
$$\Rightarrow \varrho^m_{\mathscr{G}_1}(p_i) = \varrho^m_{\mathscr{G}_2}(p_i), \sigma^n_{\mathscr{G}_1}(p_i) = \sigma^n_{\mathscr{G}_2}(p_i), \pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) = \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)$$
$$\Rightarrow \varrho_{\mathscr{G}_1}(p_i) = \varrho_{\mathscr{G}_2}(p_i), \sigma_{\mathscr{G}_1}(p_i) = \sigma_{\mathscr{G}_2}(p_i).$$
$$\Rightarrow \mathscr{G}_1 = \mathscr{G}_2.$$

(iii) Easy and left to the readers.
(iv) If $\mathscr{G}_1 \Subset \mathscr{G}_2 \Subset \mathscr{G}_3$, then $\forall p_i \in \mathbb{P}$ we have $0 \leq \varrho_{\mathscr{G}_1}(p_i) \leq \varrho_{\mathscr{G}_2}(p_i) \leq \varrho_{\mathscr{G}_3}(p_i) \leq 1$ and $1 \geq \sigma_{\mathscr{G}_1}(p_i) \geq \sigma_{\mathscr{G}_2}(p_i) \geq \sigma_{\mathscr{G}_3}(p_i) \geq 0$. It implies that $0 \leq \varrho^m_{\mathscr{G}_1}(p_i) \leq \varrho^m_{\mathscr{G}_2}(p_i) \leq \varrho^m_{\mathscr{G}_3}(p_i) \leq 1$ and $1 \geq \sigma^n_{\mathscr{G}_1}(p_i) \geq \sigma^n_{\mathscr{G}_2}(p_i) \geq \sigma^n_{\mathscr{G}_3}(p_i) \geq 0$. Thus we have

$$|\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)| \leq |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_3}(p_i)|,$$
$$|\varrho^m_{\mathscr{G}_2}(p_i) - \varrho^m_{\mathscr{G}_3}(p_i)| \leq |\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_3}(p_i)|,$$
$$|\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)| \leq |\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_3}(p_i)|,$$
$$|\sigma^n_{\mathscr{G}_2}(p_i) - \sigma^n_{\mathscr{G}_3}(p_i)| \leq |\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_3}(p_i)|,$$
$$|\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)| \leq |\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_3}(p_i)|,$$
$$|\pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_3}(p_i)| \leq |\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_3}(p_i)|.$$

Therefore we have

$$d^L_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2) = \frac{1}{3rln2}\sum_{1}^{r} ln \begin{pmatrix} (1+|\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_2}(p_i)|) \\ (1+|\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_2}(p_i)|) \\ (1+|\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)|) \end{pmatrix}$$
$$\leq \frac{1}{3rln2}\sum_{1}^{r} ln \begin{pmatrix} (1+|\varrho^m_{\mathscr{G}_1}(p_i) - \varrho^m_{\mathscr{G}_3}(p_i)|) \\ (1+|\sigma^n_{\mathscr{G}_1}(p_i) - \sigma^n_{\mathscr{G}_3}(p_i)|) \\ (1+|\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - \pi^{\frac{m+n}{2}}_{\mathscr{G}_3}(p_i)|) \end{pmatrix}$$
$$= d^L_{F^n_m}(\mathscr{G}_1, \mathscr{G}_3).$$

Similarly, $d^L_{F^n_m}(\mathscr{G}_2, \mathscr{G}_3) \leq d^L_{F^n_m}(\mathscr{G}_1, \mathscr{G}_3)$.

Theorem 3.5. *Let $\mathscr{G}_1, \mathscr{G}_2 \in \mathcal{F}^n_m(\mathbb{P})$. Then:*

(i) $d^{L}_{F^n_m}(\mathscr{G}^c_1, \mathscr{G}^c_2) = d^{L}_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2)$.

(ii) $d^{L}_{F^n_m}(\mathscr{G}_1, \mathscr{G}^c_2) = d^{L}_{F^n_m}(\mathscr{G}^c_1, \mathscr{G}_2)$.

(iii) $d^{L}_{F^n_m}(\mathscr{G}_1, \mathscr{G}^c_1) = 0 \Leftrightarrow \varrho_{\mathscr{G}_1}(p_i) = \sigma_{\mathscr{G}_1}(p_i), \quad \forall 1 \leq i \leq r$.

Proof. Follows from definitions 3.3 and 2.3.

3.2. Tangent inverse distance measure for (m,n)-FSs

Definition 3.6. *Let* $\mathbb{P} = \{p_1, p_2, ...p_r.\}$ *and* $\mathscr{G}_1, \mathscr{G}_2 \in \mathcal{F}^n_m(\mathbb{P})$. *The (m,n)- fuzzy tangent inverse distance* $d^{TI}_{F^n_m}$ *between* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *is defined as follows:*

$$d^{TI}_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2) = \frac{1}{3r}\sum_{1}^{r}\begin{pmatrix} |tan^{-1}\varrho^m_{\mathscr{G}_1}(p_i) - tan^{-1}\varrho^m_{\mathscr{G}_2}(p_i)| \\ +|tan^{-1}\sigma^n_{\mathscr{G}_1}(p_i) - tan^{-1}\sigma^n_{\mathscr{G}_2}(p_i)| \\ +|tan^{-1}\pi^{\frac{m+n}{2}}_{\mathscr{G}_1}(p_i) - tan^{-1}\pi^{\frac{m+n}{2}}_{\mathscr{G}_2}(p_i)| \end{pmatrix}. \quad (4)$$

Theorem 3.7. *The (m,n)- fuzzy tan inverse distance measure* $d^{TI}_{F^n_m}$ *between two(m,n)-FSs defined in definition 3.6 is a valid distance measure.*

Theorem 3.8. *Let* $\mathscr{G}_1, \mathscr{G}_2 \in \mathcal{F}^n_m(\mathbb{P})$. *Then:*

(i) $d^{TI}_{F^n_m}(\mathscr{G}^c_1, \mathscr{G}^c_2) = d^{TI}_{F^n_m}(\mathscr{G}_1, \mathscr{G}_2)$.

(ii) $d^{TI}_{F^n_m}(\mathscr{G}_1, \mathscr{G}^c_2) = d^{TI}_{F^n_m}(\mathscr{G}^c_1, \mathscr{G}_2)$.

(iii) $d^{TI}_{F^n_m}(\mathscr{G}_1, \mathscr{G}^c_1) = 0 \Leftrightarrow \varrho_{\mathscr{G}_1}(p_i) = \sigma_{\mathscr{G}_1}(p_i), \quad \forall 1 \leq i \leq r$.

Proof. Follows from definitions 3.6 and 2.3.

4. Applications of (m,n)-Fuzzy Distance Measures in Pattern Recognition

In pattern recognition, determining the degree of similarity or difference between data objects plays a pivotal role. Classical approaches often struggle to cope with ambiguous or imprecise data. To address this, (m,n)-fuzzy distance measures have been proposed as a flexible alternative that incorporates both membership and non-membership information.

These measures are particularly advantageous in fuzzy classification scenarios, where patterns may not have clearly defined boundaries. For example, in tasks such as face recognition or medical image classification, overlapping features between categories are common. Using (m,n)-fuzzy distances allows for a graded comparison of pattern features, making classification decisions more robust and informed. The integration of (m,n)-fuzzy distances into clustering algorithms, such as fuzzy c-means or its variants, has shown to enhance grouping effectiveness. These measures provide finer granularity when calculating similarity, which is essential in datasets with overlapping or noisy attributes.

Additionally, (m,n)-fuzzy distance measures have practical utility in other domains such as document classification, gesture recognition, and fault detection, where uncertain and vague data must be handled accurately. Their capability to model uncertainty makes them a suitable tool for real-world recognition systems.

Example 4.1. (**Categorization of Structural Substances**) Assume there are five pre-characterized construction substances represented via $(4, 6)$-fuzzy sets, denoted as $\mathscr{A}_j$ for $j = 1, \ldots, 5$. These materials are structured across a descriptive attribute set $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, as shown in Table 1. Additionally, we are given a test sample $\mathscr{A}$ that needs to be identified with one of the known material classes.

From Table 2, we observe that among all calculated (m, n)-fuzzy distance values, the smallest is consistently found between $\mathscr{A}_2$ and $\mathscr{A}$ across all four distinct measures. This indicates high resemblance, graphically substantiated in Figure 1, where $\mathscr{A}_2$ yields the lowest dissimilarity scores. Therefore, $\mathscr{A}$ can be logically classified as part of the same group as $\mathscr{A}_2$, validating the competence of the fuzzy distance methodology for accurate material classification.

Table 1: $(4, 6)$-fuzzy representation of material features

Attribute	$\mathscr{A}_1$	$\mathscr{A}_2$	$\mathscr{A}_3$	$\mathscr{A}_4$	$\mathscr{A}_5$	$\mathscr{A}$
ω_1	(0.5, 0.8)	(0.6, 0.7)	(0.3, 0.4)	(0.5, 0.3)	(0.4, 0.7)	(0.7, 0.6)
ω_2	(0.6, 0.4)	(0.7, 0.3)	(0.7, 0.5)	(0.4, 0.4)	(0.2, 0.6)	(0.8, 0.2)
ω_3	(0.8, 0.3)	(0.6, 0.2)	(0.9, 0.3)	(0.6, 0.2)	(0.5, 0.4)	(0.4, 0.3)
ω_4	(0.6, 0.9)	(0.8, 0.6)	(0.4, 0.8)	(0.4, 0.7)	(0.5, 0.3)	(0.7, 0.8)
ω_5	(0.1, 0.4)	(0.3, 0.5)	(0.2, 0.3)	(0.2, 0.6)	(0.4, 0.2)	(0.4, 0.2)

Table 2: Computed dissimilarities between known and unknown samples

Distance Metric	$(\mathscr{A}_1, \mathscr{A})$	$(\mathscr{A}_2, \mathscr{A})$	$(\mathscr{A}_3, \mathscr{A})$	$(\mathscr{A}_4, \mathscr{A})$	$(\mathscr{A}_5, \mathscr{A})$
$d^{nH}_{F^n_m}$	0.2177	**0.1551**	0.2643	0.2618	0.3058
$d^{nE}_{F^n_m}$	0.2010	**0.1445**	0.2262	0.2536	0.2423
$d^{L}_{F^n_m}$	0.2002	**0.1111**	0.2137	0.1911	0.1858
$d^{TI}_{F^n_m}$	0.1525	**0.0816**	0.1647	0.1450	0.1421

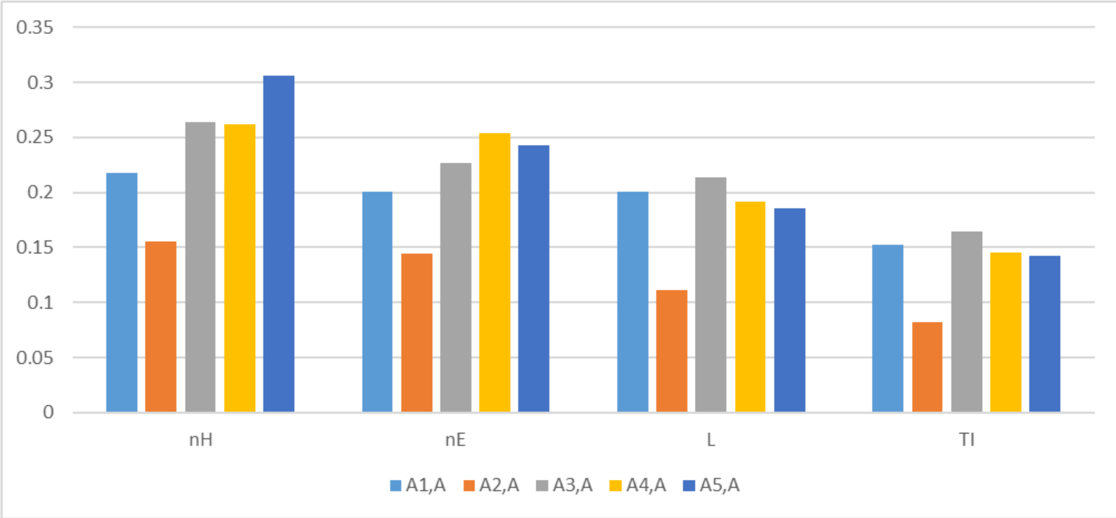

Figure 1: Visual comparison of dissimilarities for construction materials

Example 4.2. (**Identification of Plant Leaf Conditions**) Plant health is critical to both natural and agricultural systems, yet leaf-related diseases can disrupt productivity significantly. This illustration utilizes fuzzy distance evaluation to determine the likely disease based on observed symptoms. Let $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$ be a set of observed symptoms: ψ_1 = Dark brown leaf, ψ_2 = Brown leaf, ψ_3 = Yellow leaf, ψ_4 = Patches, and ψ_5 = Spots. Each disease category is modeled by $(6, 10)$-refined orthopair fuzzy sets as $\mathscr{A}_k$ for $k = 1, \ldots, 5$, corresponding to conditions such as Gray Leaf Spot, Bacterial Canker, etc., with membership data in Table 3. A new unknown symptom profile $\mathscr{A}$ is tested for diagnosis. Table 4 outlines the resulting distances from $\mathscr{A}$ to each known condition. It is evident that $\mathscr{A}_2$ exhibits the least deviation under all four distance formulas, as also depicted in Figure 2. Thus, $\mathscr{A}$ can be inferred to match Bacterial Canker most closely, verifying the reliability of the technique.

Table 3: $(6, 10)$-fuzzy symptom for leaf disease diagnosis

Symptom	$\mathscr{A}_1$	$\mathscr{A}_2$	$\mathscr{A}_3$	$\mathscr{A}_4$	$\mathscr{A}_5$	$\mathscr{A}$
ψ_1	(0.45, 0.95)	(0.25, 0.75)	(0.95, 0.55)	(0.85, 0.45)	(0.15, 0.95)	(0.35, 0.70)
ψ_2	(0.95, 0.35)	(0.85, 0.25)	(0.35, 0.85)	(0.65, 0.45)	(0.25, 0.65)	(0.80, 0.30)
ψ_3	(0.95, 0.65)	(0.75, 0.35)	(0.95, 0.45)	(0.15, 0.95)	(0.95, 0.15)	(0.70, 0.40)
ψ_4	(0.45, 0.65)	(0.15, 0.95)	(0.85, 0.15)	(0.45, 0.75)	(0.95, 0.55)	(0.20, 0.90)
ψ_5	(0.55, 0.95)	(0.15, 0.85)	(0.55, 0.35)	(0.95, 0.15)	(0.55, 0.95)	(0.25, 0.80)

Table 4: (m,n)-Fuzzy distances between symptom profile and diseases

Distance Metric	$(\mathscr{A}_1, \mathscr{A})$	$(\mathscr{A}_2, \mathscr{A})$	$(\mathscr{A}_3, \mathscr{A})$	$(\mathscr{A}_4, \mathscr{A})$	$(\mathscr{A}_5, \mathscr{A})$
$d^{nH}_{F^n_m}$	0.2193	**0.0643**	0.4737	0.4442	0.3935
$d^{nE}_{F^n_m}$	0.2316	**0.0668**	0.4387	0.4141	0.3793
$d^{L}_{F^n_m}$	0.3939	**0.0970**	0.3268	0.3477	0.4188
$d^{TI}_{F^n_m}$	0.3114	**0.0716**	0.2555	0.2711	0.3305

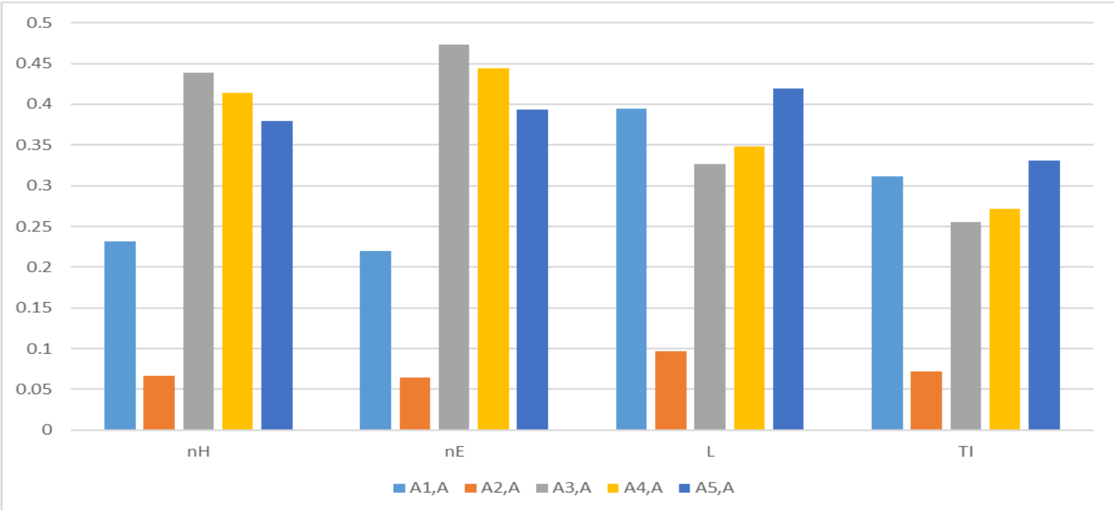

Figure 2: Graph showing (m, n)-fuzzy distances for disease identification

5. Conclusion

The (m,n)-FS is a highly effective generalization of fuzzy structures that is well-suited for addressing uncertainty and imprecision in decision-making problems. With its m and n parameters, the (m,n)-FS is capable of accommodating a broader range of information than IFS, PFS, FFS, and n-ROFS for $n \geq 3$. This paper presents logarithmic and tangent inverse distance measures, for (m,n)-FSs. The distance measures established for (m,n)-FS information include those for IFS, PFS, FFS and n-ROFS information as special cases. To assess their effectiveness, we apply our proposed distance measures to and building material problems and plant leaf disease classification compare them with existing distance measure for (m,n)-FSs. Graphical representations are provided to represent the accuracy, reliability and effectiveness of the established measures. Our findings indicate that our defined distance measures are more appropriate and generalizable for real-world problems than existing measures.

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