South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 1 (2025), pp. 127-134

DOI: 10.56827/SEAJMMS.2025.2101.11 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ON R_L TOPOLOGICAL SPACES

S. Jafari, A. Pigazzini and R. Pinčák*

Mathematical and Physical Science Foundation
4200 Slagelse, DENMARK

E-mail : saeidjafari@topositus.com, pigazzini@topositus.com

*Institute of Experimental Physics,
Slovak Academy of Sciences,
Watsonova 47, 043 53 Košice, SLOVAK REPUBLIC

E-mail : pincak@saske.sk

(**Received:** *Jul. 15, 2024* **Accepted:** *Apr. 18, 2025* **Published:** *Apr. 30, 2025)*

Abstract: The aim of this paper is to introduce a new separation axiom called R_L and study some of its fundamental properties.

Keywords and Phrases: L-bounded set, R_L-separation axiom, countably compact.

2020 Mathematics Subject Classification: 54B05, 54C08; Secondary: 54D05.

1. Preliminaries

The notion of R_0 topological space is introduced by Shanin [16] in 1943. Davis [4] rediscovered it independently and studied some properties of this weak separation axiom. Several topologists (e. g. [8], [9], [10], [13]) further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1 but strictly weaker than T_2.

Throughout the paper (X, τ) (or simply X) will always denote a topological space. For a subset A of X, the closure and interior of A in X are denoted by $Cl(A)$ and $Int(A)$, respectively. Recall that a topological space (X, τ) is said to be an R_0 space [4] if every open set contains the closure of each of its singletons. A topological space (X, τ) is said to be R_1 [4] if for x, y in X with $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $Cl(\{x\})$ is a subset of U and $Cl(\{y\})$ is a subset of V. For the interest of the reader it should be mentioned that regularity implies R_1 and R_1 implies R_0.

A topological space (X, τ) is said to be Kolmogorov or T_0 if for each pair of two distinct points, there exists an open set containing one of them but not the other. A topological space (X, τ) is said to be T_1 if for each pair of distinct points x and y of X, there exists a pair of open sets one containing x but not y and the other containing y but not x. A topological space (X, τ) is said to be Hausdorff or T_2 if for each pair of distinct points x and y of X, there exists a pair of disjoint open sets such that one containing x and the other containing y. A subset G of a topological space (X, τ) is called a difference set (briefly D-set) [17] if there are two open sets U and V of X such that $U \neq X$ and $G = U \setminus V$. A topological space (X, τ) is said to be D_0 if for any distinct pair of points x and y of X, there exists a D-set of X containing x but not y or a D-set of X containing y but not x. A topological space (X, τ) is said to be countably compact if every countable open cover of X admits a finite subcover. Recall that a topological space is Alexandroff [1] if arbitrary intersection of open sets are open.

In what follows, we refer the interested reader to [12] for the basic definitions and notations. Recall that a representation of a C^*-algebra $\mathcal{A}$ consists of a Hilbert space $\mathcal{H}$ and a *-morphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$is the C^*-algebra of bounded operators on $\mathcal{H}$. A subspace $\mathcal{I}$ of a C^*-algebra $\mathcal{A}$ is called a primitive ideal if $\mathcal{A}= ker(\pi)$ for some irreducible representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$. The set of all primitive ideals of a $C^\star$-algebra $\mathcal{A}$ plays a very important role in noncommutative spaces and its relation to particle physics. We denote this set by Prim $\mathcal{A}$. As Landi [12] mentions, for a noncommutative C^*-algebra, there is more than one candidate for the analogue of the topological space X:

1. The structure space of $\mathcal{A}$ or the space of all unitary equivalence classes of irreducible *-representations and
2. The primitive spectrum of $\mathcal{A}$ or the space of kernels of irreducible *-representations which is denoted by Prim $\mathcal{A}$. Observe that any element of Prim $\mathcal{A}$ is a two-sided *-ideal of $\mathcal{A}$.

It should be noticed that for a commutative C^*-algebra, 1 and 2 are the same but this is not true for a general C^*-algebra $\mathcal{A}$. Natural topologies can be defined on spaces of 1 and 2. But here we are interested in the Jacobsen (or hull-kernel) topology defined on Prim $\mathcal{A}$ by means of closure operators. The interested reader may refer to [5] for basic properties of Prim $\mathcal{A}$.

Proposition 1.1. [5] *The space Prim $\mathcal{A}$ is a T_0-space.*

Davis in [4] proved that T_1-spaces are precisely those which are both R_0 and T_0. For the convenience of the reader, we bring the meaning of Hasse diagram from [12]:

A pictorial representation of the topology of a poset is obtained by constructing the associated Hasse diagram: one arranges the points of the poset at different levels and connects them by using the following rules :
1. if $x \prec y$, then x is at a lower level than y;
2. if $x \prec y$ and there is no z such that $x \prec z \prec y$, then x is at the level immediately below y and these two points are connected by a link. Now we have the following result whose proof is due to Professor Giovanni Landi in a private communication to the first author for more than 15 years ago. He referred to a concrete example in his book [12], i.e. the second Hasse diagram, the singleton x_1 is open but its closure is made of the points $\{x_1, x_3, x_4\}$. This shows that Prim $\mathcal{A}$ is not an R_0-space.

Proposition 1.2. *The space Prim $\mathcal{A}$ is not an R_0-space.*

Remark 1.3. *Since T_1-spaces are precisely those which are both R_0 and T_0, therefore the space Prim $\mathcal{A}$ can not be T_1. Indeed if the space Prim $\mathcal{A}$ was R_0, then the closures of the points will partition it but this is impossible, and also whenever a point belonged to an open set, say U, then there could not exist a closed set F with $x \in F \subseteq U$. Moreover in such space which is not R_0, a normal space can not be completely regular, see (Corollary 3.1,* [7]*). According to Tong* [17]*, a topological space (X, τ) is D_0 if and only if it is T_0. Hence the space Prim $\mathcal{A}$ is also D_0. Recall that a topological space (X, τ) is called symmetric if for x and y in X, $x \in Cl(\{y\})$ implies $y \in Cl(\{x\})$. It is obvious that the space Prim $\mathcal{A}$ can not be symmetric.*

Remark 1.4. *It is worth-noticing that Kanjamapornkul and Pinčák* [11] *obtained some interesting and important results when studying time series data. They showed that one can find a time series data in spinor field* [3] *whose underlying structure is Kolmogorov in time series data. They also discussed the situation when a time series is considered in terms of quaternionic projective space* [2]*. They proved that a new space of time series is a Kolmogorov space with eight hidden dimensions with spin invariant property in time series data by which it relates to quantum entaglement qubit state* [15] *in time series data. Here is the main result of their paper (Theorem 1,* [11]*) which they showed that a space of financial time series data is a covering space S^7 with based space in $X \simeq \mathbb{H}P^1$, where $\mathbb{H}P^1$ is the quaternionic projective space. It is a Kolmogorov space, i.e., it is a space satisfying the T_0-separation axiom. It definitely can not be T_1 and thus not R_0.*

Definition 1. *A topological space (X, τ) is said to be R_1 if for every x, y in X with $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $Cl(\{x\})$ is a subset of U and $Cl(\{y\})$ is a subset of V.*

Proposition 1.5. ([4]) *If (X, τ) is a R_1 topological space, then (X, τ) is R_0.*

Definition 2. *Let (X, τ) and (Y, σ) be topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*

(1) irresolute [14] *if the preimage of every open subset of Y is open in X,*

(2) open if the image of every open subset of X is open in Y.

Definition 3. *A subset A of a topological space (X, τ) is said to be L-bounded* [6] *in X if it is contained in some countable union of the members of every open covering of X.*

2. R_L topological spaces

Definition 4. *If A and B are subsets of a topological space (X, τ) and A is L-bounded in X, by $A <<_L B$ we mean that $A \subset B$ and A is L-bounded in the subspace (B, τ_B) of X.*

In the following, let $b_L(X)$ denotes the L-bounded subsets of X.

Definition 5. *A topological space (X, τ) is said to be R_L if the following condition holds*

$A <<_L U$ *implies* $Cl(A) \subset U$ *for all* $A \in b_L(X)$ *and* $U \in \tau$.

A regular countably compact space is an example of R_L topological spaces.

Proposition 2.1. *Every R_L topological space is R_0.*

Proof. Suppose x is an arbitrary point of X and U is an arbitrary non-empty open set such that $x \in U$. Then $\{x\} <<_L U$ which implies $Cl(\{x\}) \subset U$.

Lemma 2.2. *Let (X, τ) be a R_1 Alexandroff topological space. Let $A \in b_L(X)$ and $D \subset X$ such that $A <<_L D$. If $x \in X$, $Cl(x) \cap D = \emptyset$, then x and A have disjoint open-neighbourhoods.*

Proof. By the fact that X is R_1, for each $x \in X$ and $y \in D$, there are disjoint neighborhoods U_x and V_y of x and y, respectively. We have $D \subset \cup\{V_y \mid y \in D\}$ which implies the existence of a countable subcover of A, say $V_{y_1}, V_{y_2}, \ldots, V_{y_k}$ such that $A \subset \cup_{i=1}^{k} V_{y_i} = V$. By setting $U = \cap_{i=1}^{k} U_{x_i}$ which is open. This implies that $U \cap V = \emptyset$.

By using the above lemma one can prove the following result.

Proposition 2.3. *Every R_1 Alexanderoff topological space is R_L.*

Proof. Let $A \in b_L(X)$, $U \in \tau$ such that $A <<_L U$. We prove that $Cl(A) \subset U$.

Assume that $a \in Cl(A)$. Let $Cl(\{a\}) \cap U = \emptyset$. By Lemma 2.2, there exists disjoint open neighborhoods V_a, V_A of a and A respectively. Hence $a \notin Cl(A)$ which is a contradiction. Therefore, $Cl(\{a\}) \cap U \neq \emptyset$. Let $x \in Cl(\{a\}) \cap U$. Since X is R_1, we have that $x \in Cl(\{a\})$ which implies $a \in Cl(\{x\})$. So $x \in U$ implies $Cl(\{x\}) \subset U$. Hence $a \in Cl(\{x\}) \subset U$. This means that $Cl(A) \subset U$.

Theorem 2.4. *Every subspace of a R_L topological space is R_L.*
Proof. Let (X, τ) be R_L and $Y \subset X$. Since every subset of a L-bounded set is L-bounded, let $A \subset Y$ such that $A \in b_L(Y)$ and suppose $U \in \tau_Y$. Assume that $A <<_L U$. Then there exists $V \in \tau$ such that $U = V \cap Y$. We show that $Cl(A) \subset U$. We have $A \in b_L(X)$. Thus, $A <<_L V$. Then $Cl(A)_X \subset V$ in X. Therefore, $Cl(A) \cap Y \subset V \cap Y$ in X. Thus, $Cl(A) \subset U$ in Y and hence the result.

Next, we will prove that product of finitely many R_L spaces is a R_L space. In order to prove this, we need the following lemma.

Lemma 2.5. *Let (X, τ) and (Y, σ) be two topological spaces. If $A \times B$ is a L-bounded set in the product space $X \times Y$, then both A and B are L-bounded sets in X and Y, respectively.*
Proof. Assume $A \times B$ is a L-bounded set in $X \times Y$. Let $\{O_\alpha : \alpha \in I\}$ be a open cover of Y. Then $B \subset \bigcup\{O_\alpha : \alpha \in I\}$. So for any $a \in A$, $\{a\} \times B \subset A \times B$ is L-bounded in $X \times Y$. Thus, $B \cong \{a\} \times B \subset \bigcup\{\{a\} \times O_\alpha : \alpha \in I\} \subset \{a\} \times Y \subset X \times Y$. Therefore, $B \cong \{a\} \times B \subset \bigcup_{i=1}^{n}(\{a\} \times O_{\alpha_i}) \cong \bigcup_{i=1}^{n} O_{\alpha_i}$. So B is a L-bounded set in Y.

Similarly, we can show that A is a L-bounded set in X.

Now we are ready to prove that product of finitely many R_L spaces is a R_L space.

Theorem 2.6. *Finite products of R_L topological spaces is R_L.*
Proof. Let (X, τ) and (Y, σ) be two R_L topological spaces. We claim that the product space $X \times Y$ is R_L. Let $A \times B \in b_L(X \times Y), U \times V \in \tau \times \sigma$ such that $A \times B <<_L U \times V$. We show that $Cl(A \times B) \subset U \times V$. Using Lemma 2.5, A is in $b_L(X)$ and B is in $b_L(Y)$. Also, $A <<_L U$ and $B <<_L V$. Then we have $Cl(A \times B) = Cl(A) \times Cl(B) \subset U \times V$.

Using induction, we can show that finite products of R_L spaces is R_L.

Lemma 2.7. *If a function $f : (X, \tau) \to (Y, \sigma)$ is open and bijective, then $f^{-1}(Cl(B)) \subset Cl(f^{-1}(B))$ for each subset B of Y.*
Proof. Suppose that $x \notin Cl(f^{-1}(B))$. Then there exists a open set U in X containing x such that $U \cap f^{-1}(B) = \emptyset$; hence $f(U) \cap B = \emptyset$. Since $f(U)$ is open, $f(U) \cap Cl(B) = \emptyset$ and $U \cap f^{-1}(Cl(B)) = \emptyset$. Therefore, we have $x \notin f^{-1}(Cl(B))$.

This shows that $f^{-1}(Cl(B)) \subset Cl(f^{-1}(B))$.

Theorem 2.8. *Let (X, τ) be countably compact and a function $f : (X, \tau) \to (Y, \sigma)$ be a continuous, open surjection. If (X, τ) is R_L, then (Y, σ) is R_L.*

Proof. Suppose that (X, τ) is R_L. Let $B \in b_L(Y), V \in \sigma$ such that $B <<_L V$. We show that $Cl(B) \subset V$. Since (X, τ) is countably compact, we have $f^{-1}(B) \in b_L(X)$. Since f is continuous, $f^{-1}(V) \in \tau$ such that $f^{-1}(B) <<_L f^{-1}(V)$. Since (X, τ) is R_L, $Cl(f^{-1}(B)) \subset f^{-1}(V)$. Since f is open, by Lemma 2.7 we have $f^{-1}(Cl(B)) \subset Cl(f^{-1}(B)) \subset f^{-1}(V)$. Since f is surjective, we obtain $Cl(B) \subset f(Cl(f^{-1}(B))) \subset V$. This shows that (Y, σ) is R_L.

Theorem 2.9. *Let a function $f : (X, \tau) \to (Y, \sigma)$ be an open, irresolute injection. If (Y, σ) is R_L, then (X, τ) is R_L.*

Proof. Suppose that (Y, σ) is R_L. Let $A \in b_L(X), U \in \tau$ such that $A <<_L U$. We show that $Cl(A) \subset U$. Let $\{V_\alpha : \alpha \in \Delta\}$ be any open cover of Y. Then, since f is irresolute, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a open cover of X. Since $A \in b_L(X)$, there exists a finite subfamily Δ_0 of Δ such that $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$ and hence $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\}$. Therefore, we obtain $f(A) \in b_L(Y)$. Since f is open, $f(U) \in \sigma$ such that $f(A) <<_L f(U)$. Since (Y, σ) is R_L, we have $Cl(f(A)) \subset f(U)$. Since f is injective, we obtain $A = f^{-1}(f(A)) \subset f^{-1}(Cl(f(A))) \subset f^{-1}(f(U)) = U$. Since f is irresolute, $f^{-1}(Cl(f(A)))$ is closed and hence $Cl(A) \subset U$. This shows that (X, τ) is R_L.

References

[1] Alexandroff P., Diskrete Räume, Mat. Sb., 2(1937), 501-518.

[2] Arnol'd V. I., Relatives of the Quotient of the Complex Projective Plane by the Complex Conjugation. Tr. Mat. Inst. Steklova, Nauka, Moscow, 1999, vol. 224, ch. 5, pp. 56–67, collection of papers dedicated to the 90th anniversary of academician Lev Semenovich Pontryagin. Algebra. Topology. Differential equations and their applications.

[3] Cohen R. L., The Topology of Fiber Bundles Lecture Notes, 1998, Standford University.

[4] Davis A. S., Indexed systems of neighborhoods for general topological spaces, Amer. Math. Monthly, 68(1961), 886-893.

[5] Dixmier J., Les C^*-algébres et leurs représentations, Gauthiier-Villars, 1964.

[6] Dontchev J., Ganster M. and Rose D., Ideal resolvability, Topology Appl., 93(1) (1999), 1-16.

[7] Dorsett C., R_0 and R_1 topological spaces, Mat. Vesnik, 2(15)(30) (1078), 117-122.

[8] Dube K. K., A note on R_0 topological spaces, Mat. Vesnik, 11(1974), 203-208.

[9] Hall D. W., Murphy S. K. and Rozycki J., On spaces which are essentially T_1, J. Austr. Math. Soc., 12(1971), 451-455.

[10] Herrlich H., A concept of Nearness, Gen. Topol. Appl., 4(1974), 191-212.

[11] Kanjamapornkul K. and Pinčák R., Kolmogorov space in time series data, Math. Mathods Appl. Sci., (2016)

[12] Landi G., An introduction to noncommutative spaces and their geometry, Published in Lecture Notes in Physics: Monographs, Vol. 51, Springer-Verlag, Berlin Heidelberg 2002.

[13] Naimpally S. A., On R_0-topological spaces, Ann. Univ. Sci. Budapest Eötvös Sect. Math., 10(1967), 53-54.

[14] Reilly A. S. and Vamanamurhty M. K., On α-continuity in topological spaces, Acta. Math. Hungar., 45(-12) (1985), 27-32.

[15] Rigetti C., Mosseri R., and Devoret M., Geometric approach to digital quantum information, Quantum Information Processing, Vol. 3, No. 6 (2004), 351–380.

[16] Shanin N. A., On separation in topological spaces, Dokl. Akad. Nauk. SSSR, 38(1943), 110-113.

[17] Tong J., A separation axioms between T_0 and T_1, Ann. Soc. Sci. Bruxelles, 96 II (1982), 85-90.