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BOUNDS FOR A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS RELATED TO SHELL-LIKE CURVES ASSOCIATED WITH THE (p, q)-SALAGEAN DERIVATIVE

P. Nandini, L. Dileep* and S. Latha**

Department of Mathematics,
JSS Academy of Technical Education,
Srinivaspura Bengaluru - 560060, Karnataka, INDIA

E-mail : pnandinimaths@gmail.com

*Department of Mathematics,
Vidyavardhaka College of Engineering,
Mysuru - 570002, Karnataka, INDIA

E-mail : dileepL84@vvce.ac.in

**Department of Mathematics,
Yuvaraja's College, Mysore - 570005, Karnataka, INDIA

E-mail : drlatha@gmail.com

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Abstract: This paper aims to investigate a new subclass of bi-univalent functions defined by the (p, q)-Salagean derivative, associated with shell-like curves connected with Fibonacci numbers. It also examines the coefficient estimates and Fekete-Szegö inequalities for functions in this class.

Keywords and Phrases: Bi-univalent functions, Fekete-Szegö inequality, Fibonacci numbers, Shell-like curve and (p, q)-Salagean derivative.

2020 Mathematics Subject Classification: 30C45, 30C50.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Further, let $\mathcal{A}$ represent the class of functions analytic in $\mathbb{D}$, thus satisfying the condition:

$$f(0) = f'(0) - 1 = 0.$$

Then, each of the functions f in $\mathcal{A}$ has the following Taylor series expansion:

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Suppose $\mathcal{S}$ is a subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. Also let $\mathcal{P}$ be the class of Carathéodory functions $p : \mathbb{D} \to \mathbb{C}$ of the form $p(z) = 1 + c_1 z + c_2 z^2 + ...$, $z \in \mathbb{D}$ such that $\Re\{p(z)\} > 0$. An analytic function f is subordinate to an analytic function g in $\mathbb{D}$, written as $f \prec g$ $(z \in \mathbb{D})$, provided there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. It follows from Schwarz Lemma [7] that

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \textit{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}), \quad z \in \mathbb{D}.$$

The Koebe One-quarter theorem [5] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$.
Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions defined in the unit disc $\mathbb{D}$ given by (1.1). Note that the functions

$$f_1(z) = \frac{z}{1-z}, \quad f_2(z) = -\log(1-z), \quad f_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \quad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}, \quad f_3^{-1}(w) = \frac{e^w-1}{e^w}$$

are elements of Σ. This subject has been discussed extensively in the pioneering work by Srivastava et al. [18] who revived the study of analytic and bi-univalent functions in recent years.

It is well-known that the Fibonacci sequence denoted by $\{F_n\}$ is such that each number is the sum of the two preceding ones, starting from 0 and 1; that is $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$. It is also well-known that we can write

$$F_n = \frac{\varphi^n - \tau^n}{\sqrt{5}} = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \quad (n \in \mathbb{N}_0) \tag{1.3}$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad and \quad \tau = \frac{1-\sqrt{5}}{2} \approx -0.618. \tag{1.4}$$

Now we recall the following function

$$\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z - \tau^2 z^2}$$

introduced by Sokól in [17], where τ is given by (1.4). The function $\tilde{p}$ is not univalent in $\mathbb{D}$, but it is univalent in the disc $|z| < {}^{(3-\sqrt{5})}/_{2} \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}\left(-{}^{1}/_{2\tau}\right) = 1$ and $\tilde{p}\left(e^{\mp i arcos(1/4)}\right) = {}^{\sqrt{5}}/_{5}$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfills the golden section. In [11], taking $\tau z = t$, Raina and Sokól showed that

$$\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z - \tau^2 z^2} = 1 + \sum_{n=1}^{\infty} (F_{n-1} + F_{n+1}) \tau^n z^n,$$

where Fibonacci number F_n given by (1.3) and $\tau = \dfrac{1-\sqrt{5}}{2}$. These researchers also found that

$$\tilde{p}(z) = 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + \ldots.$$

For $0 < q < p \leq 1$, the Jackson (p,q)-derivative of a function $f \in \mathcal{A}$ is given by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad (z \neq 0).$$

Therefore for f as in (1.1), we have

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1},$$

where $[n]_{p,q} = \dfrac{p^n - q^n}{p-q}, \quad (0 < q < p \leq 1).$

Recently for $f \in \mathcal{A}$, Ahuja [1] defined and discussed the (p,q)-Salagean differential operator as given below:

$$\mathcal{S}^0_{p,q}f(z) = f(z)$$

$$\mathcal{S}^1_{p,q}f(z) = zD_{p,q}f(z)$$

$$\mathcal{S}^m_{p,q}f(z) = zD_{p,q}(\mathcal{S}^{m-1}_{p,q}f(z)), \quad (m \in N_0, z \in \mathbb{D}).$$

For f of the form (1.1), we get

$$\mathcal{S}^m_{p,q}f(z) = z + \sum_{n=2}^{\infty} [n]^m_{p,q} a_n z^n,$$

Further for functions g of the form (1.2), we define

$$\mathcal{S}^m_{p,q}g(w) = w - a_2[2]^m_{p,q}w^2 + (2a_2^2 - a_3)[3]^m_{p,q}w^3 + ...$$

Motivated by works of Ahuja et al. [2], Dziok et al. [6], Güney et al. [9], we define a new subclass of bi-univalent functions related to shell-like curves associated with (p,q)-Salagean derivative.

Definition 1.1. *For $0 < q < p \leq 1$ and $0 \leq \alpha \leq 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{SLM}_{\alpha,\Sigma}(p,q,m,\tilde{p}(z))$ if it satisfies the following subordinations:*

$$\alpha \frac{D_{p,q}(\mathcal{S}^{m+1}_{p,q}f(z))}{D_{p,q}(\mathcal{S}^m_{p,q}f(z))} + (1-\alpha)\frac{\mathcal{S}^{m+1}_{p,q}f(z)}{\mathcal{S}^m_{p,q}f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z - \tau^2 z^2}, \tag{1.5}$$

and

$$\alpha \frac{D_{p,q}(\mathcal{S}^{m+1}_{p,q}g(w))}{D_{p,q}(\mathcal{S}^m_{p,q}g(w))} + (1-\alpha)\frac{\mathcal{S}^{m+1}_{p,q}g(w)}{\mathcal{S}^m_{p,q}g(w)} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w - \tau^2 w^2}, \tag{1.6}$$

where $\tau = {(1-\sqrt{5})}/{2} \approx -0.618$, $g = f^{-1}$ *given by (1.2) and* $z, w \in \mathbb{D}$.

Definition 1.2. *For* $0 < q < p \leq 1$ *and* $\alpha = 0$, *a function* $f \in \Sigma$ *is said to be in the class* $\mathcal{SLM}_{\Sigma}(p, q, m, \tilde{p}(z))$ *if it satisfies the following subordinations:*

$$\frac{\mathcal{S}_{p,q}^{m+1} f(z)}{\mathcal{S}_{p,q}^{m} f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}, \tag{1.7}$$

and

$$\frac{\mathcal{S}_{p,q}^{m+1} g(w)}{\mathcal{S}_{p,q}^{m} g(w)} \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \tag{1.8}$$

where $\tau = {(1-\sqrt{5})}/{2} \approx -0.618$, $g = f^{-1}$ *given by (1.2) and* $z, w \in \mathbb{D}$.

Definition 1.3. *For* $0 < q < p \leq 1$ *and* $\alpha = 1$, *a function* $f \in \Sigma$ *is said to be in the class* $\mathcal{KLM}_{\alpha,\Sigma}(p, q, m, \tilde{p}(z))$ *if it satisfies the following subordinations:*

$$\frac{D_{p,q}(\mathcal{S}_{p,q}^{m+1} f(z))}{D_{p,q}(\mathcal{S}_{p,q}^{m} f(z))} = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}, \tag{1.9}$$

and

$$\frac{D_{p,q}(\mathcal{S}_{p,q}^{m+1} g(w))}{D_{p,q}(\mathcal{S}_{p,q}^{m} g(w))} = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \tag{1.10}$$

where $\tau = {(1-\sqrt{5})}/{2} \approx -0.618$, $g = f^{-1}$ *given by (1.2) and* $z, w \in \mathbb{D}$.

Remark 1.4.

(i) $\mathcal{SLM}_{0,\Sigma}(p, q, 0, \tilde{p}(z)) = \mathcal{SL}_{\Sigma}(p, q, \tilde{p}(z))$ *and* $\mathcal{SLM}_{1,\Sigma}(p, q, 0, \tilde{p}(z)) = \mathcal{KSL}_{\Sigma}(p, q, \tilde{p}(z))$, *the classes of bi-univalent functions studied by Nandini and Latha* [8].

(ii) $\mathcal{SLM}_{\alpha,\Sigma}(1, 1, n, \tilde{p}(z)) = \mathcal{SLM}_{\alpha,\Sigma}(n, \tilde{p}(z))$, *the class of bi-univalent functions established by Gurmeet Singh and Gagandeep Singh* [16].

(iii) $\mathcal{SLM}_{\alpha,\Sigma,}(1, q, 0, \tilde{p}(z)) = \mathcal{SLM}_{\Sigma}(q, \alpha)$, $\mathcal{SLM}_{0,\Sigma}(1, q, 0, \tilde{p}(z)) = q - \mathcal{SL}_{\Sigma}$ *and* $\mathcal{SLM}_{1,\Sigma}(1, q, 0, \tilde{p}(z)) = q - \mathcal{KSL}_{\Sigma}$, *the classes of bi univalent functions studied by Ahuja* [2].

(iv) $\mathcal{SLM}_{\alpha,\Sigma}(1, 1, 0, \tilde{p}(z)) = \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$, $\mathcal{SLM}_{0,\Sigma}(1, 1, 0, \tilde{p}(z)) = \mathcal{SL}_{\Sigma}(\tilde{p}(z))$ *and* $\mathcal{SLM}_{1,\Sigma}(1, 1, 0, \tilde{p}(z)) = \mathcal{KL}_{\Sigma}(\tilde{p}(z))$, *the classes of bi-univalent functions defined by Güney* [9].

In order to prove our results we need the following lemma.

Lemma 1.5. [3, 4] *If* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1 z + c_2 z^2 + ...$, *then*

$$|c_n| \leq 2, \quad n \geq 1.$$

2. Initial Coefficient Estimates and Fekete-Szego Inequalities

Theorem 2.1. *For* $0 < q < p \leq 1$, $0 \leq \alpha \leq 1$, *Let* $f \in \mathcal{SLM}_{\alpha,\Sigma}(p, q, m, \tilde{p}(z))$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{|(\eta - \psi)\tau + (1 - 3\tau)\zeta|}}, \tag{2.1}$$

$$|a_3| \leq \frac{|\tau| \left\{|(\eta - \psi)\tau + (1 - 3\tau)\zeta| + \eta|\tau|\right\}}{\eta|(\eta - \psi)\tau + (1 - 3\tau)\zeta|}, \tag{2.2}$$

for any real number μ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{\eta}, & |\mu - 1| \leq \dfrac{|\tau(\eta - \psi) + (1 - 3\tau)\zeta|}{|\tau|\eta} \\ \dfrac{|\mu - 1||\tau|^2}{|\tau(\eta - \psi) + (1 - 3\tau)\zeta}, & |\mu - 1| \geq \dfrac{|\tau(\eta - \psi) + (1 - 3\tau)\zeta|}{|\tau|\eta}, \end{cases} \tag{2.3}$$

where

$$\eta = [3]_{p,q}^m([3]_{p,q} - 1)\left[1 + \alpha([3]_{p,q} - 1)\right], \tag{2.4}$$

$$\psi = [2]_{p,q}^{2m}([2]_{p,q} - 1)\left[1 + \alpha([2]_{p,q}^2 - 1)\right], \tag{2.5}$$

$$\zeta = [2]_{p,q}^{2m}([2]_{p,q} - 1)^2\left[1 + \alpha([2]_{p,q} - 1)\right]^2. \tag{2.6}$$

The result is sharp.

Proof. As $f \in \mathcal{SLM}_{\alpha,\Sigma}(p, q, m, \tilde{p}(z))$, so by Definition 1.1 and using the principle of subordination, there exists Schwarz functions $u, v : \mathbb{D} \to \mathbb{D}$ with $u(0) = 0 = v(0)$, such that

$$\alpha \frac{D_{p,q}(\mathcal{S}_{p,q}^{m+1} f(z))}{D_{p,q}(\mathcal{S}_{p,q}^m f(z))} + (1 - \alpha)\frac{\mathcal{S}_{p,q}^{m+1} f(z)}{\mathcal{S}_{p,q}^m f(z)} = \tilde{p}(u(z)) \tag{2.7}$$

and

$$\alpha \frac{D_{p,q}(\mathcal{S}_{p,q}^{m+1} g(w))}{D_{p,q}(\mathcal{S}_{p,q}^m g(w))} + (1 - \alpha)\frac{\mathcal{S}_{p,q}^{m+1} g(w)}{\mathcal{S}_{p,q}^m g(w)} = \tilde{p}(v(w)). \tag{2.8}$$

Now define the function,

$$h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots.$$

Then

$$\tilde{p}(u(z)) = 1 + \frac{c_1}{2}\tau z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2} + \frac{3c_1^2}{2}\tau\right)\tau z^2 + \dots. \tag{2.9}$$

Similarly we define the function,

$$k(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w + d_2 w^2 + d_3 w^3 + ...$$

Then

$$\tilde{p}(v(w)) = 1 + \frac{d_1}{2}\tau w + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2} + \frac{3d_1^2}{2}\tau\right)\tau w^2 + ... \tag{2.10}$$

and by considering the LHS of (2.7) and (2.8), we have

$$\alpha\frac{D_{p,q}(\mathcal{S}_{p,q}^{m+1}f(z))}{D_{p,q}(\mathcal{S}_{p,q}^{m}f(z))} + (1-\alpha)\frac{\mathcal{S}_{p,q}^{m+1}f(z)}{\mathcal{S}_{p,q}^{m}f(z)}$$

$$\begin{aligned}=&1 + [2]_{p,q}^m([2]_{p,q}-1)\left(1+\alpha([2]_{p,q}-1)\right)a_2 z + \big\{[3]_{p,q}^m([3]_{p,q}-1)\\ &\left(1+\alpha([3]_{p,q}-1)\right)a_3 - [2]_{p,q}^{2m}([2]_{p,q}-1)\left(1+\alpha([2]_{p,q}^2-1)\right)a_2^2\big\}z^2 + ...\end{aligned}$$

and

$$\alpha\frac{D_{p,q}(\mathcal{S}_{p,q}^{m+1}g(w))}{D_{p,q}(\mathcal{S}_{p,q}^{m}g(w))} + (1-\alpha)\frac{\mathcal{S}_{p,q}^{m+1}g(w)}{\mathcal{S}_{p,q}^{m}g(w)}$$

$$\begin{aligned}=&1 - [2]_{p,q}^m([2]_{p,q}-1)\left(1+\alpha([2]_{p,q}-1)\right)a_2 w + \big\{2[3]_{p,q}^m([3]_{p,q}-1)\left(1+\alpha([3]_{p,q}-1)\right.\\ &+[2]_{p,q}^{2m}(1-[2]_{p,q})(1+\alpha([2]_{p,q}^2-1))\big)a_2^2 - [3]_{p,q}^m([3]_{p,q}-1)\left(1+\alpha([3]_{p,q}-1)\right)a_3\big\}w^2\\ &+ . \quad . \quad .\end{aligned}$$

Using (2.9), (2.10) and the above two equations in (2.7) and (2.8) and equating the coefficients of z, z^2, w and w^2 we get

$$[2]_{p,q}^m([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))a_2 = \frac{c_1}{2}\tau, \tag{2.11}$$

$$\begin{aligned}[3]_{p,q}^m([3]_{p,q}-1)\left(1+\alpha([3]_{p,q}-1)\right)a_3 - [2]_{p,q}^{2m}([2]_{p,q}-1)\left(1+\alpha([2]_{p,q}^2-1)\right)a_2^2\\ = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{3c_1^2}{4}\tau^2,\end{aligned} \tag{2.12}$$

$$-[2]_{p,q}^m([2]_{p,q-1})(1+\alpha([2]_{p,q}-1))a_2 = \frac{d_1}{2}\tau, \tag{2.13}$$

and

$$\begin{aligned}&\left\{2[3]_{p,q}^{m}([3]_{p,q}-1)\left(1+\alpha([3]_{p,q}-1)+[2]_{p,q}^{2m}(1-[2]_{p,q})(1+\alpha([2]_{p,q}^{2}-1))\right)\right\}a_2^2\\&-\left\{[3]_{p,q}^{m}([3]_{p,q}-1)\left(1+\alpha([3]_{p,q}-1)\right)\right\}a_3=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{3d_1^2}{4}\tau^2.\end{aligned} \tag{2.14}$$

From (2.11) and (2.13), we have

$$c_1=-d_1 \tag{2.15}$$

and also

$$2[2]_{p,q}^{2m}\left([2]_{p,q}-1\right)^2\left(1+\alpha([2]_{p,q}-1)\right)^2a_2^2=\frac{(c_1^2+d_1^2)\tau^2}{4} \tag{2.16}$$

$$a_2^2=\frac{(c_1^2+d_1^2)\tau^2}{8[2]_{p,q}^{2m}([2]_{p,q}-1)^2(1+\alpha([2]_{p,q}-1))^2}. \tag{2.17}$$

Adding (2.12) and (2.14), we get

$$\begin{aligned}&2\left\{[3]_{p,q}^{m}([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))-[2]_{p,q}^{2m}([2]_{p,q}-1)(1+\alpha([2]_{p,q}^{2}-1))\right\}a_2^2\\&=\frac{1}{2}(c_2+d_2)\tau-\frac{1}{4}(c_1^2+d_1^2)\tau+\frac{3}{4}(c_1^2+d_1^2)\tau^2.\end{aligned} \tag{2.18}$$

Using (2.16) in the above equation, we get

$$4a_2^2=\frac{(c_2+d_2)\tau^2}{[(\eta-\psi)\tau+(1-3\tau)\zeta]}, \tag{2.19}$$

where η,ψ and ζ are given by (2.4), (2.5) and (2.6) respectively.

On application of Lemma 1.5 and the triangular inequality we get the required inequality for $|a_2|$.

To find $|a_3|$ first we subtract (2.14) from (2.12) and then by using (2.15), we get

$$2[3]_{p,q}^{m}([3]_{p,q}-1)\left[1+\alpha([3]_{p,q}-1)\right](a_3-a_2^2)=\frac{1}{2}(c_2-d_2)\tau$$

$$a_3=\frac{(c_2-d_2)\tau}{4[3]_{p,q}^{m}([3]_{p,q}-1)\left[1+\alpha([3]_{p,q}-1)\right]}+a_2^2. \tag{2.20}$$

Now by using (2.19) in (2.20) and Lemma 1.5, we get the coefficient bound for $|a_3|$. From (2.20), we have

$$a_3-\mu a_2^2=\frac{(c_2-d_2)\tau}{4([3]_{p,q}-1)[1+\alpha([3]_{p,q}-1)]}+(1-\mu)a_2^2. \tag{2.21}$$

By substituting (2.17) in (2.21), we have

$$a_3-\mu a_2^2=\frac{(c_2-d_2)\tau}{4[3]_{p,q}^m([3]_{p,q}-1)[1+\alpha([3]_{p,q}-1)]}+(1-\mu)\left(\frac{(c_2+d_2)\tau^2}{4[(\eta-\psi)\tau+(1-3\tau)\zeta]}\right)$$

$$=\left(h(\mu)+\frac{\tau}{4[3]_{p,q}^m\left(([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))\right)}\right)c_2+\left(h(\mu)-\frac{\tau}{4[3]_{p,q}^m\left(([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))\right)}\right)d_2 \tag{2.22}$$

where

$$h(\mu)=\frac{(1-\mu)\tau^2}{4\left((\eta-\psi)\tau+(1-3\tau)\zeta\right)}.$$

Thus by taking modulus of (2.22) and using Lemma 1.5, we conclude that

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{\eta}, & |\mu-1|\leq\frac{|\tau|}{4\eta}\\ 4|h(\mu)|, & |\mu-1|\geq\frac{|\tau|}{4\eta}.\end{cases}.$$

Using the above equation we can get the desired bound for the Fekete-Szegö problem. We exhibit the sharpness by defining $f(z)$ as

$$\alpha\frac{D_{p,q}(\mathcal{S}_{p,q}^{m+1}f(z))}{D_{p,q}(\mathcal{S}_{p,q}^{m}f(z))}+(1-\alpha)\frac{\mathcal{S}_{p,q}^{m+1}f(z)}{\mathcal{S}_{p,q}^{m}f(z)}=\tilde{p}(z).$$

Corollary 2.2. *For* $0<q<p\leq 1$, *Let* $f\in\mathcal{SLM}_\Sigma(p,q,m,\tilde{p}(z))$. *Then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{\left|\left([3]_{p,q}^m([3]_{p,q}-1)-[2]_{p,q}^{2m}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m}([2]_{p,q}-1)^2\right|}},$$

$$|a_3|\leq\frac{|\tau|\left\{\left|\left([3]_{p,q}^m([3]_{p,q}-1)-[2]_{p,q}^{2m}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m}([2]_{p,q}-1)^2\right|+[3]_{p,q}^m([3]_{p,q}-1)|\tau|\right\}}{[3]_{p,q}^m([3]_{p,q}-1)\left|\left([3]_{p,q}^m([3]_{p,q}-1)-[2]_{p,q}^{2m}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m}([2]_{p,q}-1)^2\right|},$$

for any real number μ ,

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{[3]_{p,q}^m([3]_{p,q}-1)}, & |\mu-1|\leq\frac{\left|\left([3]_{p,q}^m([3]_{p,q}-1)-[2]_{p,q}^{2m}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m}([2]_{p,q}-1)^2\right|}{[3]_{p,q}^m([3]_{p,q}-1)|\tau|}\\ \frac{|\mu-1||\tau|^2}{\left|\left([3]_{p,q}^m([3]_{p,q}-1)-[2]_{p,q}^{2m}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m}([2]_{p,q}-1)^2\right|}, & \\ & |\mu-1|\geq\frac{\left|\left([3]_{p,q}^m([3]_{p,q}-1)-[2]_{p,q}^{2m}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m}([2]_{p,q}-1)^2\right|}{[3]_{p,q}^m([3]_{p,q}-1)|\tau|}.\end{cases}$$

Corollary 2.3. *For* $0<q<p\leq 1$, *Let* $f\in\mathcal{KLM}_\Sigma(p,q,m,\tilde{p}(z))$. *Then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{\left|\left([3]_{p,q}^{m+1}([3]_{p,q}-1)-[2]_{p,q}^{2m+2}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m+2}([2]_{p,q}-1)^2\right|}},$$

$$|a_3| \leq \frac{|\tau|\left\{\left|\left([3]_{p,q}^{m+1}([3]_{p,q}-1)-[2]_{p,q}^{2m+2}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m+2}([2]_{p,q}-1)^2\right|+[3]_{p,q}^{m+1}([3]_{p,q}-1)|\tau|\right\}}{[3]_{p,q}^{m+1}([3]_{p,q}-1)\left|\left([3]_{p,q}^{m+1}([3]_{p,q}-1)-[2]_{p,q}^{2m+2}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m+2}([2]_{p,q}-1)^2\right|},$$

for any real number μ ,

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{|\tau|}{[3]_{p,q}^{m+1}([3]_{p,q}-1)}, & |\mu-1| \leq \frac{\left|\left([3]_{p,q}^{m+1}([3]_{p,q}-1)-[2]_{p,q}^{2m+2}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m+2}([2]_{p,q}-1)^2\right|}{[3]_{p,q}^{m+1}([3]_{p,q}-1)|\tau|} \\ \frac{|\mu-1||\tau|^2}{\left|\left([3]_{p,q}^{m+1}([3]_{p,q}-1)-[2]_{p,q}^{2m+2}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m+2}([2]_{p,q}-1)^2\right|} \\ \quad |\mu-1| \geq \frac{\left|\left([3]_{p,q}^{m+1}([3]_{p,q}-1)-[2]_{p,q}^{2m+2}([2]_{p,q}-1)\right)\tau+(1-3\tau)[2]_{p,q}^{2m+2}([2]_{p,q}-1)^2\right|}{[3]_{p,q}^{m+1}([3]_{p,q}-1)|\tau|} \end{cases}$$

Remark 2.4. *For* $m=0, \alpha=0$ *and* $m=0, \alpha=1$, *Theorem 2.1 gives the coefficient estimates and Fekete-Szegö inequalities for the classes* $\mathcal{SL}_{\Sigma}(p, q.\tilde{p}(z))$ *and* $\mathcal{SL}_{\sigma}(p, q, \tilde{p}(z))$ *respectively studied by Nandini and Latha* [8].

For $p=1$ and $q \to 1$ we obtain the following results due to Gurmeet Singh and Gagandeep Singh [16].

Corollary 2.5. *If* $f \in \mathcal{SLM}_{\alpha,\Sigma}(m, \tilde{p}(z))$, *then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4^m(1+\alpha)^2+[2(1+2\alpha)3^m-(3\alpha^2+9\alpha+4)4^m]\tau}},$$

$$|a_3| \leq \frac{|\tau|4^m[(1+\alpha)^2-(3\alpha^2+9\alpha+4)\tau]}{2(1+2\alpha)3^m[4^m(1+\alpha)^2+(2(1+2\alpha)3^m-(3\alpha^2+9\alpha+4)4^m)\tau]}$$

and

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)3^m}, & |\mu-1| \leq \frac{[(2(1+2\alpha)3^m-(3\alpha^2+9\alpha+4)4^m)\tau+(1+\alpha)^2 4^m]}{2|\tau|(1+2\alpha)3^m} \\ \frac{|1-\mu|\tau^2}{[(2(1+2\alpha)3^m-(3\alpha^2+9\alpha+4)4^m)\tau+(1+\alpha)^2 4^m]}, & |\mu-1| \geq \frac{[(2(1+2\alpha)3^m-(3\alpha^2+9\alpha+4)4^m)\tau+(1+\alpha)^2 4^m]}{2|\tau|(1+2\alpha)3^m} \end{cases}$$

For $p=1$ and $m=0$ we obtain the following results due to Ahuja [2].

Corollary 2.6. *For* $q \in (0,1)$, $\alpha \in [0,1]$*and* $\mu \in \mathbb{R}$, *let* $f \in \mathcal{SLM}_{\Sigma}(q, \alpha)$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{|\tau(\kappa-\chi)+(1-3\tau)\xi|}}$$

$$|a_3| \leq \frac{|\tau|\{|\tau(\kappa-\chi)+(1-3\tau)\xi|+|\tau|\kappa\}}{\kappa|\tau(\kappa-\chi)+(1-3\tau)\xi|}$$

and

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{|\tau|}{\kappa}, & |\mu-1| \leq \frac{|\tau[\kappa-\chi]+(1-3\tau)\xi|}{|\tau|\kappa} \\ \frac{|\mu-1||\tau|^2}{|\tau[\kappa-\chi]+(1-3\tau)\xi|}, & |\mu-1| \geq \frac{|\tau[\kappa-\chi]+(1-3\tau)\xi|}{|\tau|\kappa}, \end{cases}$$

where $\kappa = ([3]_q - 1)(1 + \alpha([3]_q - 1))$, $\chi = ([2]_q - 1)(1 + \alpha([2]_q^2 - 1))$ *and*

$$\xi = ([2]_q - 1)^2(1 + \alpha([2]_q - 1))^2.$$

Remark 2.7. *For* $p = 1, m = 0$, $\alpha = 0$ *and* $p = 1, m = 0, \alpha = 1$, *Theorem 2.1 gives the coefficient estimates and Fekete-Szegö inequalities for the classes* $q - \mathcal{SL}_\Sigma$ *and* $q - \mathcal{KSL}_\Sigma$ *respectively defined by Ahuja* [2].

For $p = 1$, $q = 1$ and $m = 0$ we obtain the following results due to Güney [9].

Corollary 2.8. *Let* f *given by (1.1) be in the class* $\mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{(1+\alpha)^2 - (1-\alpha)(2+3\alpha)\tau}},$$

$$|a_3| \leq \frac{|\tau|[(1+\alpha)^2 - (3\alpha^2 + 9\alpha + 4)\tau]}{2(1+2\alpha)(1+\alpha)[(1+\alpha) - (2+3\alpha)\tau]}$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{2(1+2\alpha)}, & |\mu - 1| \leq \frac{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}{2(1+2\alpha)|\tau|} \\ \frac{|1-\mu|\tau^2}{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}, & |\mu - 1| \geq \frac{(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]}{2(1+2\alpha)|\tau|} \end{cases}.$$

Remark 2.9. *For* $p = 1, q = 1, m = 0$, $\alpha = 0$ *and* $p = 1, q = 1, m = 0$, $\alpha = 1$, *Theorem 2.1 gives Coefficient estimates and Fekete-Szegö inequalities for the function classes* $\mathcal{SL}_\Sigma(\tilde{p}(z))$ *and* $\mathcal{KL}_\Sigma(\tilde{p}(z))$ *respectively defined by Güney* [9].

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