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COMPARING HYBRID FUNCTIONS AND HAAR WAVELETS TO ESTABLISH QUADRATURE RULES FOR NUMERICAL INTEGRATION

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Abstract: A quadrature rule based on hybrid functions and uniform Haar wavelets is provided to find approximations of the values of definite integrals. The main advantages of this approach are its simplicity and efficacy. We offer error estimates and numerical examples to verify the convergence and accuracy of the suggested approach.

Keywords and Phrases: Numerical method, Quadrature rule, Haar wavelets, Hybrid functions.

2020 Mathematics Subject Classification: 65D30, 65D32, 65GXX.

1. Introduction

In science and engineering, numerical integration has numerous uses. Regarding the quadrature rule of numerical integration, a great deal of research has been done in this field. Polynomial interpolation serves as the foundation for the quadrature rule. To determine the weights associated with nodes, interpolating polynomials

are utilized. There are several disadvantages to numerical quadrature. These include:
i. In the case of the Newton-Cotes quadrature rule, using a large number of identically spaced nodes may result in chaotic behaviour with high degree polynomial interpolation.
ii. The Gaussian quadrature rule also uses polynomial interpolation as its foundation, but it chooses its nodes and weights to optimize the final rule's accuracy.

Newton-Cotes formulas (trapezoidal, Simpson's rule) and Gaussian quadrature are two methods used in quadrature methods to approximate definite integrals with greater accuracy. The use of computational tools such as SciPy, MATLAB, and Mathematica facilitates implementation, and error analysis (Runge's phenomenon, Richardson extrapolation) and adaptive techniques enhance accuracy. Special function-based quadrature techniques, like the Lobatto and Chebyshev methods, increase efficiency even more.

The method of undetermined coefficients can be used to derive the Gaussian quadrature rule, but the resulting equations for the 2 n unknown nodes and weights are nonlinear. The process is laborious when done by hand, and nodes and weights must be tabulated before integrals can be evaluated numerically.

The number of coefficients grows exponentially in higher dimensions, and the method's computational cost rises significantly as well. The current method is based on taking one integral at a time and using the hybrid function or Haar wavelet method for a single integral in order to prevent the growing computational cost. The same procedure is used to evaluate other integrals in a similar way after one integral has been solved.

We learned a variety of techniques to solve integral problems in calculus and engineering mathematics classes, as detailed in [13]. These techniques included the change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, and others. Different types of generalized two-variable Bessel functions are examined in the article, with a focus on how they relate to second-order Bessel-type differential equations. It simplifies the study of Bessel function properties by introducing the Bessel operator through shift operators. In the current article, we primarily examine the following three categories of integrals, for which it is difficult to find solutions using the previously listed techniques.

$$\int \mathrm{e}^{ax} e^{\lambda e e^{bx}} dx \tag{1}$$

$$\int \mathrm{e}^{ax} \sin\left(\lambda e^{bx}\right) dx \tag{2}$$

$$\int \mathrm{e}^{ax} \cos\left(\lambda e^{bx}\right) dx \tag{3}$$

Where a, b, λ are real numbers, and $b, \lambda \neq 0$. The study used a combination of research methods, including manual calculations to find solutions and Matlab verification of those solutions.

The study uses Kampé de Fériet-type Hermite polynomials to extend first and second kind Chebyshev polynomials, as in [4]. It derives integral representations and establishes links with Gegenbauer polynomials. In [7] Chebyshev and Block Pulse Wavelet are applied to solve integrals where as in [1, 2, 5, 6, 10, 11], a novel approach based on hybrid functions and Haar wavelets is used to numerically integrate double and triple integrals with variable limits. This method is an improvement and generalization of our previous method. This method offers several advantages over the conventional quadrature rule. The new approach is tested on a number of benchmark problems. When comparing the two methods, the hybrid functions method produces better results than the Haar wavelets method. It is shown that the present approach is both easier to use and more accurate than the hybrid functions method and the symmetric Gauss Legendre quadrature.

The present study suggests a novel approach based on hybrid functions and basic Haar wavelets. These are the benefits of this approach:
i. Offers a more precise solution than the current approach.
ii. A built-in method is used to determine optimal weights in terms of wavelets or hybrid function coefficients. With the new method, we can find the ideal weights without consulting a number of tables.
iii. The collocation points serve as nodal points and no quadrature nodes are required.
iv. The new approach does not require solving a nonlinear system resulting from the unknown nodes and weights because it computes the integrals explicitly.
v. Direct application that is easy to understand and doesn't require the use of any intermediary techniques.

2. Numerical Methods based on Quadrature rules

a. Haar wavelets

For the family of haar wavelets specified in the range [a,b], the scaling function is

$$h_1(x) = \begin{cases} 1 \text{ for } x \in [a, b) \\ 0 \qquad\qquad\qquad \text{elsewhere} . \end{cases} \tag{4}$$

Additionally defined on the interval [a,b) and provided by, the mother wavelet for the family of Haar wavelets.

$$h_2(x) = \begin{cases} 1 \text{ for } x \in a, \frac{a+b}{2}) \\ -1 \text{ for } x \in \left[\frac{a+b}{2}, b\right) \\ 0 \quad \text{elsewhere} . \end{cases} \tag{5}$$

The dilation and translation processes used to create $h_2(x)$ produce all the other functions in the Haar wavelet family, which are specified on subintervals of [a,b). With the exception of the scaling function, all functions defined for $x \in [a, b)$ in the Haar wavelets family can be represented as

$$h_i(x) = \begin{cases} 1 \text{ for } x \in [\alpha, \beta) \\ -1 \text{ for } x \in [\beta, \gamma) \\ 0 \quad \text{elsewhere} . \end{cases} \tag{6}$$

Where

$$\alpha = a + (b-a)\frac{k}{m}, \quad \beta = a + (b-a)\frac{k+0.5}{m}, \gamma = a + (b-a)\frac{k+1}{m}, \text{i} = 3, 4, .., 2\text{M}$$

The integer $m = 2^j$, where $j = 0, 1, ..J, J = 2^M$ and integer $j = 0, 1, .., m-1$. The translation parameter is denoted by the number k, whereas the integer j represents the wavelet's level. Integer J represents the highest level of resolution. $I = m+k+1$ is the formula for the relationship between I, m, and k. Because of this, the Haar wavelet functions are orthogonal to one another.

$$\int_a^b h_j(x)h_k(x)dx = \begin{cases} (b-a)2^{-j} \text{ when } j = k \\ 0 \quad \text{when } j \neq k \end{cases} \tag{7}$$

Consequently, any function $f(x)$ that can be squarely integrated in the interval $[a, b)$ can be represented as an infinite sum of Haar wavelets.

$$f(x) = \sum_{i=1}^{\alpha} a_i h_i(x) \tag{8}$$

If $f(x)$ is piecewise constant or can be approximated as piecewise constant throughout each subinterval, then the aforementioned series ends at finite terms.

b. Method of numerical integration based on Haar wavelets

The numerical integration of single integrals using Haar wavelets is discussed in this section. We consider the integral

$$\int_a^b f(x)dx \tag{9}$$

Over the stretch $[a, b]$ Using Haar wavelets, the function $f(x)$ can be approximated as

$$f(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{10}$$

By raising the value of M, the Haar wavelets approximation quickly converges to the precise function.

Lemma 2.1. [9] *The approximate value of the integral is*

$$\int_a^b f(x)dx \approx a_1(b-a)$$

It is clear from that Haar approximation involves only one coefficient in the evaluation of the definite integral. To calculate the Haar coefficient a1 we consider the nodal points

$$x_k = a + \frac{(b-a)(k-0.5)}{2M}, k = 1, 2, .., 2M \tag{11}$$

(10) Can be expressed as the discretized form, which is

$$f\left(x_k\right) = \sum_{i=1}^{2M} a_i h_i\left(x_k\right) \tag{12}$$

Lemma 2.2. [9] *The solution of the system (12) for* a_1 *is*

$$a_1 = \frac{1}{2M} \sum_{i=1}^{2M} f\left(x_k\right)$$

Consequently, we obtain the following formula for numerical integration using the quadrature method and Haar wavelets:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2M} \sum_{i=1}^{2M} f\left(a + \frac{(b-a)(k-0.5)}{2M}\right) \tag{13}$$

c. Hybrid functions

The orthogonal set of hybrid functions $\varphi_{ij}(x), i = 1, 2, ..n$ and $j = 0, 1, \ldots m-1$ is defined in the interval $[0, 1)$ Where the orders of the Legendre polynomials and the block-pulse functions are, respectively, n and m . To distinguish hybrid functions from Haar wavelets, use the notation φ. Recursively, the Legendre polynomials can be computed as

$$l_0(x) = 1, l_1(x) = x \ldots l_{k+1}(x) = \frac{2k+1}{k+1} x l_k(x) - \frac{k}{k+1} l_{k-1}(x), k = 1, 2, 3$$

Any function $f(x)$ which is square integral in the interval $[0, 1)$ can be expressed as

$$f(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \varphi_{ij}(x), \quad i, j = 1, 2, \ldots \infty, x \in [0, 1)$$

d. Method of numerical integration based on hybrid functions

Consider the case where $m = 1$,for this value of m the formula is given by

$$\int_0^1 f(x)dx \approx \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{(2i-1)}{2n}\right) \tag{16}$$

The numerical integration formulas using hybrid functions for the integral equation (9) making the substitution $x = a + (b-a)y$ were derived for different values of m in [9].i.e

For m = 1,

$$\int_a^b f(x)dx \approx \frac{(b-a)}{n} \sum_{i=1}^{n} f\left(a + \frac{(b-a)(2i-1)}{2n}\right) \tag{17}$$

For m = 2,

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2n} \sum_{i=1}^{n} \left[f\left(a + \frac{(b-a)(4i-3)}{4n}\right) + f\left(a + \frac{(b-a)(4i-1)}{4n}\right)\right] \tag{18}$$

For m = 3,

$$\int_a^b f(x)dx \approx \frac{(b-a)}{8n} \sum_{i=1}^{n} \left[3f\left(a + \frac{(b-a)(6i-5)}{6n}\right) + 2f\left(a + \frac{(b-a)(6i-3)}{6n}\right)\right.$$
$$\left. +3f\left(a + \frac{(b-a)(6i-3)}{6n}\right)\right] \tag{19}$$

For m=4,

$$\int_a^b f(x)dx \approx \frac{(b-a)}{48n}\sum_{i=1}^{n}\left[13f\left(a+\frac{(b-a)(8i-7)}{8n}\right)+11f\left(a+\frac{(b-a)(8i-5)}{8n}\right)\right.$$
$$\left.+11f\left(a+\frac{(b-a)(8i-3)}{8n}\right)+13f\left(a+\frac{(b-a)(8i-1)}{8n}\right)\right] \tag{20}$$

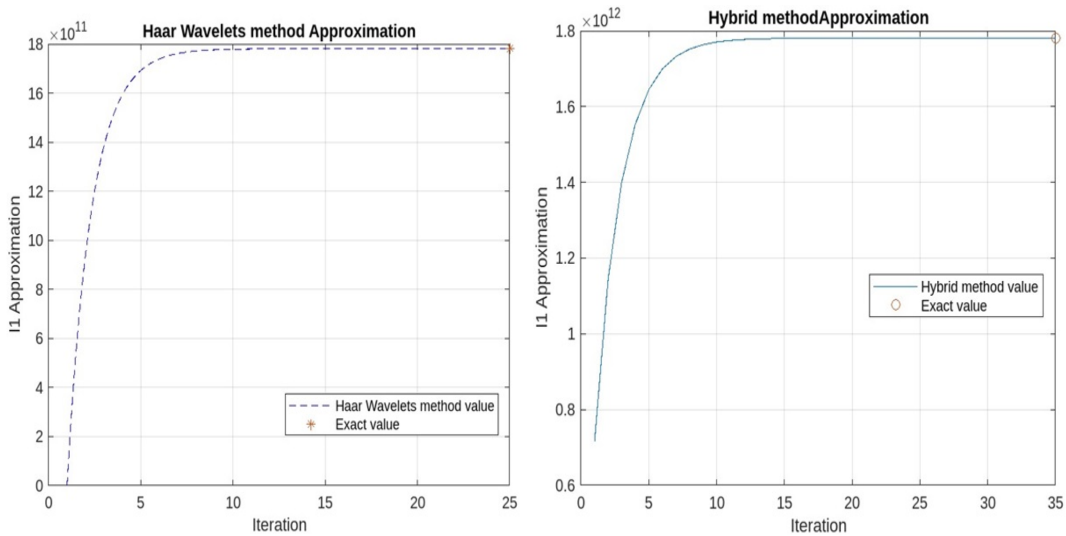
In a similar way the formulas for $m = 5, 6, 7\ldots$ given in [8].

3. Numerical Examples

In order to solve complex problems where traditional methods might have trouble with discontinuities, numerical integration is essential. The localized nature of Haar wavelets allows them to handle such cases well, whereas hybrid functions combine wavelets with orthogonal polynomials to improve accuracy. The effectiveness of these methods in solving numerical integral problems is illustrated in this study through examples.

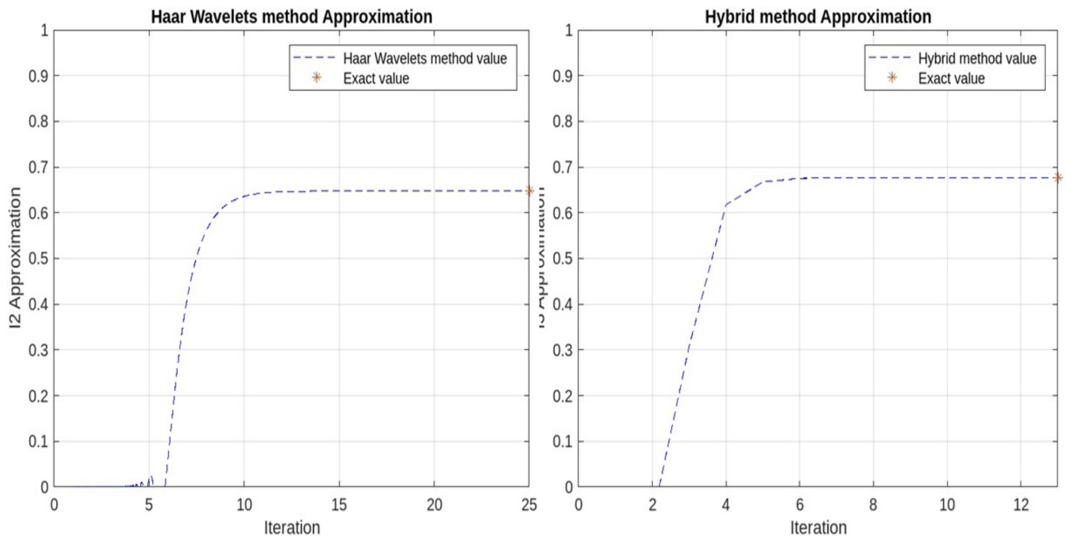
Example 3.1.

$$I_1 = \int_{-5}^{-2} \mathrm{e}^{-6x} e^{5e^{4x}} dx$$
$$= 1.781079124724728443 2.10^{12} (\text{ Chii } - \text{ Huei Yu., [13] })$$



Example 3.2.

$$I_2 = \int_{-1}^{5} \mathrm{e}^{2x} \sin\left(7e^{-6x}\right) dx$$
$$= 0.6475912563205659980 61 \text{ (Chii - Huei Yu., [13])}$$



Example 3.3.

$$I_3 = \int_{-3}^{2} \mathrm{e}^{4x} \cos\left(8e^{-3x}\right) dx$$
$$= 737.407836945388 \text{ (Chii - Huei Yu., [13])}$$

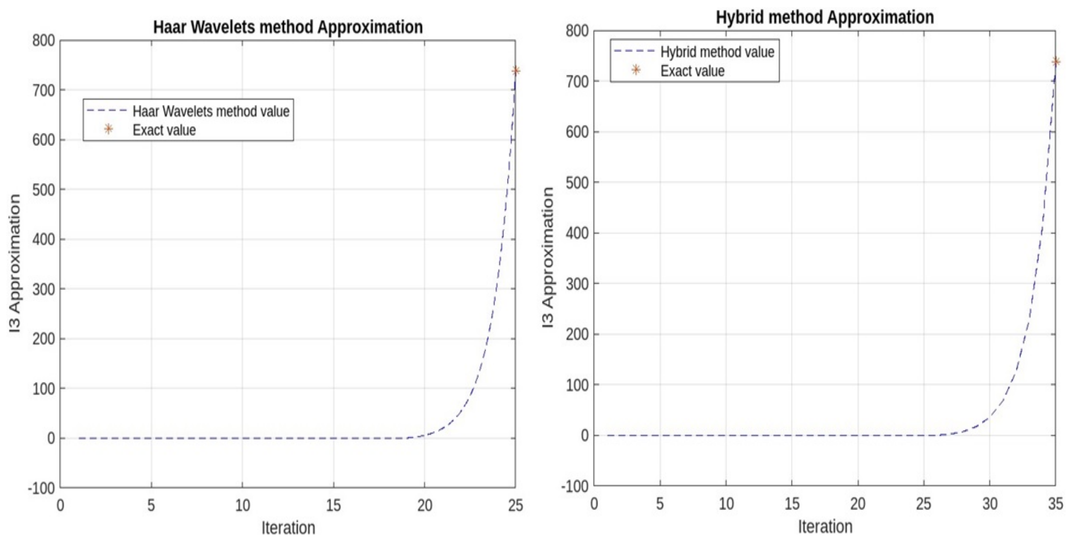


Table 1: **Comparison of numerical results with exact values for different n and m orders.**

Exact Value	Order	Haar Wavelet	Order	Hybrid Function
$I_1 = 1.7810791247247284432 \times 10^{12}$	$n = 25$	$1.781079124722314 \times 10^{12}$	$n = 35, m = 4$	$1.781070483939743 \times 10^{12}$
$I_2 = 0.6475912563205659980 68$	$n = 25$	0.647591256322499	$n = 13, m = 2$	0.646906796851105
$I_3 = 737.407836945389$	$n = 25$	$7.374078369454077 \times 10^{2}$	$n = 35, m = 4$	$7.374013465294072 \times 10^{2}$

4. Conclusion

Numerical approximations of various integral types are found by comparing hybrid functions and Haar wavelets. In the context of numerical approximation of integral equations, the straightforward applicability of Haar wavelets and the quick convergence of hybrid functions offer a strong foundation for their use.Future research might concentrate on expanding these techniques to multi-dimensional integrals and enhancing their suitability for functions with high oscillations. Furthermore, investigating adaptive mesh refinement in Haar wavelet techniques and creating hybrid function-based deep learning models for integral equations may improve their suitability for use in engineering and computational mathematics.

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