South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 1 (2025), pp. 97-102

DOI: 10.56827/SEAJMMS.2025.2101.8 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

D'ALEMBERT'S METHOD IN LAPLACE EQUATION

Eunyoung Lim, Arjun K. Rathie* and Hwajoon Kim

11458, Kyungdong Univ. Rd. 27, Yangju, Gyeonggi, SOUTH KOREA

E-mail : cellmath@gmail.com

*Vedant College of Engineering & Technology,
(Rajasthan Technical University)
Tulsi, Jakhamund, Bundi, Rajasthan, INDIA

E-mail : arjunkumarrathie@gmail.com

(**Received:** *Oct. 15, 2024* **Accepted:** *Mar. 28, 2025* **Published:** *Apr. 30, 2025*)

Abstract: D'Alembert's method is typically applied to wave equation. This study extends the existing approach by examining D'Alembert's method in Laplace equation. The value of this study lies in its various explorations of the method.

Keywords and Phrases: D'Alembert's method, Laplace equation, steady-state heat equation.

2020 Mathematics Subject Classification: 35A22, 44A10.

1. Introduction

This study began with the idea of exploring what would happen if D'Alembert's method, which is typically used to solve wave equations, were applied to the Laplace equation. While this method is not as general as the Fourier series approach, it is certainly elegant. Let us apply this elegant method to the Laplace equation and consider the potential issue. We believe there is a reason this has not been studied extensively until now. D'Alembert's method was originally used to find the solution to the vibrating string problem by setting

$$v = x + ct, w = x - ct,$$

based on the characteristic of the PDE. By differentiating these values and substituting them into the wave equation

$$u_{tt} = c^2 u_{xx},$$

where $c^2 = T/\rho$, we obtain

$$u_{vw} = 0,$$

where T is the tension and ρ is the mass of the string per unit length [6]. Solving this gives us the solution:

$$u(x,t) = \phi(x+ct) + \psi(x-ct),$$

where ϕ and ψ are arbitrary functions. Applying the initial conditions

$$u(x,0) = 0,\ u_t(x,0) = g(x),$$

we find:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Several intriguing studies have been conducted on related topics. These include the derivation of q-analogues involving even-order polynomials and q-trigonometric-type functions [1], the application of the Laplace residual power series method to solve three-dimensional fractional Helmholtz equations [2], a generalization of Kummer's quadratic transformation [3], and the investigation of fractional integral transformations of the Mittag-Leffler type E-function [9]. Initial conditions and details related to D'Alembert's method can be found in [6]. The methods for solving the Laplace equation mainly use separation of variables, Fourier transform, Green's function, finite difference method, finite element method, and physics-informed neural networks. In this paper, we will examine why D'Alembert's method, which is used to solve the wave equation, is not applicable to the Laplace equation.

Now, let us apply D'Alembert's method to the Laplace equation.

2. D'Alembert's method in Laplace equation

Consider the analytic function $f(z) = u(x,y) + iv(x,y)$. It is a well-known that the harmonic function u and v form the Laplace equation: Because,

$$u_{yy} = (-v_x)_y = (-v_y)_x = (-u_x)_x,$$

by Cauchy-Riemann equation.

Theorem 2.1. (D'Alembert's method in Laplace equation) *The solution of*

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

can be expressed as

$$u(x, y) = f(x - iy) + g(x + iy),$$

where i is the imagnary unit and u is a continuous function.

Proof. Since $u_{yy} = -u_{xx}$, let us put $f = x + iy$ and $g = x - iy$. Then

$$u_x = u_f f_x + u_g g_x = u_f + u_g$$

and so,

$$u_{xx} = (u_f + u_g)_x = (u_f + u_g)_f f_x + (u_f + u_g)_g g_x = u_{ff} + 2u_{fg} + u_{gg}$$

because u is continuous. Similarly, from $u_y = u_f f_y + u_g g_y = i(u_f - u_g)$,

$$\begin{aligned} u_{yy} &= i(u_f - u_g)_y = i(u_f - u_g)_f f_y + i(u_f - u_g)_g g_y \\ &= -(u_f - u_g)_f + (u_f - u_g)_g = -u_{ff} + 2u_{fg} - u_{gg}. \end{aligned}$$

Substituting these values into the original equation, we get

$$-u_{ff} + 2u_{fg} - u_{gg} = -u_{ff} - 2u_{fg} - u_{gg}.$$

Organizing the equality, we have $u_{fg} = 0$. This gives $u_f = h(f)$ and $u = \int h(f)df + q(g)$, where h and q are arbitrary functions of f and g, respectively. Therefore, we can simply express this as

$$u(x, y) = p(f) + q(g) = p(x + iy) + q(x - iy), \tag{1}$$

where p and q are arbitrary functions.

Now let us see if this is a solution to the given Laplace equation. This can be easily seen as a solution to the given equation by substitution and the fact that

$$i^2 = -1.$$

Since the above Laplace equation can be interpreted as a steady problem of the two-dimensional heat equation, $u_t = 0$ for t is the time. Let us consider the case where the initial temperature is 0, i.e., $u(x, 0) = 0$. Then * can be expressed as

$$u(x, y) = p(x + iy) - p(x - iy). \tag{2}$$

Example 2.2. Let us consider a Laplace equation

$$u(x,y) = a\ln(x^2+y^2)+b.$$

When the function u satisfies boundary condition $u=1$ on $(x^2+y^2)=1$ and $u=0$ on $(x^2+y^2)=e$, we can determine a and b as $a=-1,\ b=1$.
Solution. It is easy to see that this equation satisfies the Laplace equation

$$u_{xx}+u_{yy}=0,$$

and that its solution is

$$u(x,y) = -\ln(x^2+y^2)+1 = \ln\frac{e}{x^2+y^2} = (1)$$

If $u(x,0)=0$, then $x^2=e$ and so

$$u(x,y) = -\ln(e+y^2)+1 = \ln\frac{e}{e+y^2} = (2),$$

where e is the Euler's number.

Example 2.3. Let us compare (2) with the result by Laplace transform.
Solution. If we take the Laplace transform of the Laplace equation, we get

$$\begin{aligned}\pounds(u_{xx}) &= \int_0^\infty e^{-sy}u_{xx}\,dy \\ &= \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-sy}u(x,y)\,dy = \frac{\partial^2}{\partial x^2}U\end{aligned}$$

where $U=\pounds(u)$. In general, the interchanging of the integral and the derivative does not hold. However, the reason why the integral and the derivative can be swapped here is discussed in detail in [4]. Meanwhile,

$$\pounds(u_{yy}) = s^2U - su(x,0) - u_y(x,0) = s^2U.$$

Substituting these two results into the original equation, we obtain

$$\frac{\partial^2 U}{\partial x^2}+s^2U=0.$$

This gives

$$U(x,s) = q(s)cos\ sx + p(s)sin\ sx,$$

where p and q are arbitrary. From $u(x,0)=0$, $q(0)=0$. Therefore, the solution of Laplace equation has the form of

$$u(x,y) = \pounds^{-1}[q(s)cos\ sx + p(s)sin\ sx] = (2).$$

Example 2.4. Let $G = G_\alpha$ is the G_α transform, a generalized Laplace-type transform [7].
A solution of $w_t = c^2 w_{xx}$ subject to the conditions $w(0,t)=0$, $w(L,t)=0$, and $w(x,0)=f(x)$ can be represented by $w(x,t) = G^{-1}[F(x,u)]$, where

$$F(x,u) = A(u)\ (e^{\frac{-x}{c\sqrt{u}}} - e^{\frac{x}{c\sqrt{u}}})$$

$$+\frac{\sqrt{u}}{2c}u^\alpha\ (e^{\frac{-x}{c\sqrt{u}}}\int e^{\frac{x}{c\sqrt{u}}}\cdot f(x)\ dx - e^{\frac{x}{c\sqrt{u}}}\int e^{\frac{-x}{c\sqrt{u}}}f(x)\ dx). \quad [5,\ theorem\ 3]$$

In the above equation, replacing the Laplace transform with another transform only changes the form but does not affect the result. On the one hand, the reason this method is difficult to apply to the heat equation or Laplace equation is that it is challenging to implement the boundaries conditions, which do not fit well.

Acknowledgements

The corresponding author (Hj. Kim) acknowledges the support of Kyungdong University Research Fund, 2025.

References

[1] Al-Omari SK., q-analogues and properties of the Laplace-type integral operator in the quantum calculus theory, Journal of Inequalities and Applications, 8 (1) (2020), 203.

[2] Albalawi, W., Shah, R., Nonlaopon, K., El-Sherif, L.S., El-Tantawy, S. A., Laplace Residual Power Series Method for Solving Three-Dimensional Fractional Helmholtz Equations, Symmetry, 15 (2023), 194.

[3] Atia, M. J. and Rathie, A. K., On a Generalization of the Kummer's Quadratic Transformation and a Resolution of an Isolated Case, Axioms, 12(9), (2023), 821.

[4] Han Giljun and Hwajoon Kim, On the relationship between the integral of derivative and the derivative of the integral, European Journal of Pure and Applied Mathematics, 15.3 (2022), 1211-1216.

[5] Hwajoon Kim, Applications of an extended Laplace transform in heat equation, JP Journal of Heat and Mass Transfer, 19.1 (2020), 185-193.

[6] Kreyszig E., Advanced Engineering Mathematics, Wiley, Singapore, 2013.

[7] Sattaso, S., Prasertsang, P., and Prasertsang, T., Using G_α-transform to Study Higher-order Differential Equations with Polynomial Coefficients, European Journal of Pure and Applied Mathematics, 16(3) (2023), 1389-1405.

[8] Saed M. Turq, R.I. Nuruddeen, and Rab Nawaz, Recent advances in employing the Laplace homotopy analysis method to nonlinear fractional models for evolution equations and heat-typed problems, International Journal of Thermofluids, 22 (2024), 100681

[9] Sanjay B. and Faisal S. M., Fractional Integral Transformations of Mittag-Leffler Type E-function, South East Asian J, of Math. & Math. Sci., 11(1) (2015), 31-38.